மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

MANONMANIAM SUNDARANAR UNIVERSITY
TIRUNELVELI-627 012

தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

B.Sc. MATHEMATICS

II YEAR

DIFFERENTIAL EQUATIONS AND APPLICATIONS

Sub. Code: DMAM32

Prepared by

Dr. S. KALAISELVI

Assistant Professor

Department of Mathematics

Sarah Tucker College(Autonomous), Tirunelvei-7.

B.Sc. MATHEMATICS –II YEAR

DMAM32: DIFFERENTIAL EQUATIONS AND APPLICATIONS

SYLLABUS

UNIT I:

Ordinary Differential Equations: Variable separable – Homogeneous Equation – Non-Homogeneous Equations of first degree in two variables – Linear Equation – Bernoulli's Equation – Exact differential equations.

Chapter 1: Sections 1.1 – 1 . 6

UNIT II:

Equation of first order but of higher degree: Equation solvable for dy/dx - Equation solvable for y – Equation solvable for x – Clairaut's form – Linear Equations with constant coefficients: Definition – The operator D – Complete solution – Particular integrals of algebraic, exponential, trigonometric functions and their products.

Chapter 2: Sections 2.1 - 2.7

UNIT III:

Linear equations of second order: Complete solution in terms of a known integral – Reduction to normal form – Change of independent variable - Applications of first order equations: Flow of water from an orifice – Falling bodies and other rate problems, Free fall under gravity – The Brachistochrone – Fermat and Bernoulli – Simple electric circuits.

Chapter 3: Sections – 3.1 - 3.9

UNIT IV:

Partial differential equation: Formation of PDE by Eliminating arbitrary constants and arbitrary functions – Complete integral – Singular integral – General integral – Lagrange's Linear Equations.

Chapter 4: Sections – 4.1 - 4.4

UNIT V: Special methods – Standard forms.

Chapter 5: Sections - 5.1 - 5.5

Text Book:

S. Narayanan, T.K. Manicavachagom Pillay, Differential Equations And Applications, Divya Subramanian for Ananda Book Depot, Chennai,2012

DMAM32: DIFFERENTIAL EQUATIONS AND APPLICATIONS

CONTENTS

UNIT IV

UNIT V

UNIT I:

Ordinary Differential Equations: Variable separable – Homogeneous Equation – Non-Homogeneous Equations of first degree in two variables – Linear Equation – Bernoulli's Equation – Exact differential equations.

Chapter 1: Sections 1.1 – 1 . 6

Introduction:

A differential equation is an equation in which differential coefficients occurs.

Differential Equation are of two types

1. Ordinary
2. Partial

An Ordinary Differential Equations is one in which single independent variable enters, either explicitly or implicitly

Example:

1) $\frac{dy}{dx} = 2\sin x$

2) $\frac{d^2x}{dt^2} + m^2y = 0$

In (2) it must, be noted that there are & independent variables x and y and there is only one independent variable t.

A PDE is one in which at least 2 independent variables enter and the partial differential coefficients occurring in them have reference to any one of these variable

Example:

i) $x\frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = z$

$ii)\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2 = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$

The Order of differential equation is the order of the highest derivative occurring in it.

The Degree of the differential equation is the degree of the highest derivative when the differential coefficients are seared of radicals and fractions.

Example 1:

$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = a$$

Order: 2

Degree: 2

Example 2:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = x$$

Order: 2

Degree: 1

Equation of the First order and of the First Degree:

1.1. Variables Separable:

suppose an equation is of the form $f(x)dx + F(y)dy = 0$ we can directly integrate this equation and the solution is $\int f(x)dx + \int F(y)dy = c$ where c is the arbitrary constant.

Example 1:

Solve $\dfrac{dy}{dx} + \left(\dfrac{1-y^2}{1-x^2}\right)^{1/2} = 0$

Solution:

Given

$$\frac{dy}{dx} + \left(\frac{1-y^2}{1-x^2}\right)^{1/2} = 0$$

$$\frac{dy}{dx} = -\left(\frac{1-y^2}{1-x^2}\right)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = -\left[\frac{(1-y^2)^{1/2}}{(1-x^2)^{1/2}}\right]$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} = \frac{-dx}{\sqrt{1-x^2}}$$

Integrating on both sides,

$$\Rightarrow \sin^{-1} y = -\sin^{-1} x + c$$

$$\Rightarrow \sin^{-1} y + \sin^{-1} x = c$$

Example 2:

Solve $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

Solution:

$ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

$$\therefore \frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0$$

$$\Rightarrow d(x/y) + 3x^2e^{x^3}dx = 0$$

$$\Rightarrow d(x/y) + e^t dt = 0$$

Integrating on both sides,

$$\int d(x/y) + \int e^t dt = 0$$

$$x/y + e^t = \phi c$$

$$x/y + e^{x^3} = \phi c$$

Example 3:

Solve $e^x \tan y dy + (1 - e^x)\sec^2 y dy = 0$

Solution:

$$e^x \tan y dy + (1 - e^x)\sec^2 y dy = 0$$

$$e^x \tan y dy = -(1 - e^x)\sec^2 y dy$$

$$e^x \tan y dx = (e^x - 1)\sec^2 y dy$$

$$\frac{e^x dx}{e^x - 1} = \frac{\sec^2 y dy}{\tan y}$$

$$\frac{d(e^x - 1)}{e^x - 1} = \frac{d(\tan y)}{\tan y}$$

Integrating on both sides,

$\Rightarrow \log|e^x - 1| = \log|\tan y| + \log c$

taking e power on both side

$\Rightarrow e^x - 1 = c \tan y.$

Example 4:

Solve $x\sqrt{1 + y^2} + y\sqrt{1 + x^2}dy/dx = 0$

Solution:

$$x\sqrt{1 + y^2} + y\sqrt{1 + x^2}dy/dx = 0$$

$$\frac{x\sqrt{1 + y^2}dx + y\sqrt{1 + x^2}dy}{dx} = 0$$

$$x\sqrt{1 + y^2} = -y\sqrt{1 + x^2}\frac{dy}{dx}$$

$$x\sqrt{1 + y^2}dx = -y\sqrt{1 + x^2}dy$$

$$x\sqrt{1 + y^2}dx + y\sqrt{1 + x^2}dy = 0$$

$x\sqrt{1+y^2}dx = -y\sqrt{1+x^2}dy \quad \frac{xdx}{\sqrt{1+x^2}} = \frac{-ydy}{\sqrt{1+y^2}}$

$$\frac{xdx}{\sqrt{1+x^2}} = \frac{-ydy}{\sqrt{1+y^2}}$$

Take $u = 1+x^2$ $\quad v = 1+y^2$

$du = 2xdx$ $\quad dV = 2ydy$

$\frac{du}{2} = xdx$ $\quad dv/2 = y'dy$

$(1) \Rightarrow \frac{du/2}{\sqrt{u}} = \frac{-dv/2}{\sqrt{V}}$

Integrating on both sides,

$$\int u^{-1/2}du = -\int v^{-1/2}dv$$

$$\frac{u^{-1/2+1}}{-1/2+1} = -\frac{v^{-1/2+1}}{-1/2+1} + c$$

$$\frac{u^{1/2}}{1/2} = \frac{-v^{1/2}}{1/2} + c$$

$$\sqrt{1+x^2} = -\sqrt{1+y^2} + c$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

Example 5:

Solve $\tan y \ \sec^2 xdx + \tan x \sec^2 ydy = 0$

Solution:

$$\tan y \sec^2 xdx + \tan x \sec^2 ydy = 0$$

$$\tan y \sec^2 xdx = -\tan x \sec^2 ydy$$

$$\frac{\sec^2 xdx}{\tan x} = -\frac{\sec^2 ydy}{\tan y} \quad \ldots\ldots\ldots\ldots.. (1)$$

Let $u = \tan x$ $v = \tan y$

$du = \sec^2 xdx$ $dv = \sec^2 ydy$

$$\textit{From equation } (1) \ \frac{du}{u} = -\frac{dv}{v}$$

$\therefore$

$$\int \frac{du}{u} = -\int \frac{dv}{v}$$

$$\log u = -\log v + \log c$$

$$u = -v + c$$

$$\tan x + \tan y = c$$

Example 6:

Solve $\sqrt{1+x^2}dx + \sqrt{1+y^2}dy = 0$

Solution:

$\sqrt{1+x^2}dx+\sqrt{1+y^2}dy=0$

Integrating on both sides,

$$\int \sqrt{1+x^2}dx+\int \sqrt{1+y^2}dy=0$$

$$\int \sqrt{a^2+x^2}dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$$

$$\Rightarrow\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log\left(x+\sqrt{x^2+1}\right)+\frac{y}{2}\sqrt{1+y^2}+\frac{1}{2}\log\left(y+\sqrt{y^2+1}\right)=\mathrm{c}$$

$$\Rightarrow x\sqrt{1+x^2}+y\sqrt{1+y^2}+\log\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=2c$$

$$x\sqrt{1+x^2}+y\sqrt{1+y^2}+\log\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=c_1$$

Example 7:

Solve $y^2\cos\sqrt{x}dx-2\sqrt{x}e^{1/y}dy=0$

Solution:

$y^2\cos\sqrt{x}dx-2\sqrt{x}e^{1/y}dy=0$

$y^2\cos\sqrt{x}dx=2\sqrt{x}e^{1/ydy}$

$$\frac{\cos\sqrt{x}dx}{2\sqrt{x}}=\frac{e^{1/ydy}}{y^2}$$

Take

$u=\sqrt{x}$ $\quad v=1/y$

$du=\frac{1}{2}x^{-1/2}dx$ $\quad dv=-y^{-1-1}dy$

$du=\frac{1}{2\sqrt{x}}dx$ $\quad -dv=+1/y^2dy$

(1) $\Rightarrow \cos u\,du=e^v dv$

Integrating on both sides,

$$\Rightarrow\int \cos u du=-\int e^v dv$$

$\Rightarrow\sin u=-e^v+c$

$\sin\sqrt{x}+e^{1/y}=c$

Example 8:

Solve $(x-y)^2\frac{dy}{dx}=a^2$

Solution:

$(x-y)^2\frac{dy}{dx}=a^2$ ……….. (1)

Put $z=x-y$

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$$(1) \Rightarrow z^2\left(1 - \frac{dz}{dx}\right) = a^2$$

$$1 - \frac{dz}{dx} = \frac{a^2}{z^2}$$

$$1 - \frac{a^2}{z^2} = \frac{dz}{dx}$$

$$dx = \frac{dz}{\left(1 - \frac{a^2}{z^2}\right)}$$

$$dx = \frac{z^2 dz}{z^2 - a^2}$$

$$dx = \frac{(z^2 - a^2 + a^2)dz}{z^2 - a^2}$$

$$dx = \left(\frac{z^2 - a^2}{z^2 - a^2} + \frac{a^2}{z^2 - a^2}\right)dz$$

$$dx = dz + \frac{a^2}{z^2 - a^2}dz \ + c$$

Integrating on both sides,

$$\Rightarrow \int \ dx = \int \ dz + a^2 \int \ \frac{dz}{z^2 - a^2}$$

$$x = z + a^2\left[\frac{1}{2a}\log\left|\frac{z - a}{z + a}\right| + c\right]$$

$$x = x - y + \frac{a}{2}\log\left|\frac{x - y - a}{x - y + a}\right| + c$$

$$y - \frac{a}{2}\log\left|\frac{x - y - a}{x - y + a}\right| = c$$

1.2.Type B:Homogenous equation:

Consider $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} \rightarrow (1)$

where f_1 and f_2 are homogenous functions of the same degree in x and y.

$f(x,y)$ sean be written as $x^n\phi(\frac{y}{x})$ and $f_2(x,y)$ as $x^n\psi(\frac{y}{x})$

If we put $y = vx$

$$\frac{dy}{dx} = v + x$$

∴ The equation (1) becomes

$$v + x\frac{dv}{dx} = \frac{x^n\phi(v)}{x^n\psi(v)}$$

$$v + x\frac{dv}{dx} = \frac{\phi(v)}{\psi(v)}$$

The variables can be separated

$$\frac{xdv}{dx} = \frac{\phi(v)}{\psi(v)} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-(v(v) - \phi(v))}{\psi(v)}$$

$$\Rightarrow \frac{\psi(v)dv}{v(v) - \phi(v)} = \frac{-dx}{x}$$

Integrating on both sides,

$$\Rightarrow \int \frac{\psi(v)dv}{v\,\psi(v) - \phi(v)} = -\log x + c$$

$$\Rightarrow \log x + \int \frac{\psi(v)dv}{v\,\psi(v) - \phi(v)} = c$$

The solution is got by sub $\frac{y}{x}$ for v after the integration is over

Example 1:

Solve $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

Solution:

$$y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$$

$$\Rightarrow x^2\frac{dy}{dx} - xy\frac{dy}{dx} = -y^2$$

$$\frac{dy}{dx}(x^2 - xy) = -y^2$$

$$\frac{dy}{dx} = \frac{-y^2}{(x^2-xy)} \qquad \ldots\ldots\ldots\ldots (1)$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

∴ (1) becomes,

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2v^2}{x^2(v-1)}$$

$$x\frac{dv}{dx} = \frac{v^2}{v-1} - v$$

$$x\frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1}$$

$$x\frac{dv}{dx} = \frac{v}{v-1}$$

$$\frac{(v-1)dv}{v} = \frac{dx}{x}$$

$(1-\frac{1}{v})dv = \frac{dx}{x}$

Integrating on both sides,

$$\int dv - \int \frac{dv}{v} = \int \frac{dx}{x}$$

$\Rightarrow \mathrm{v} - \mathrm{logv} = \log x + c$

$\log x + \log v - v = -c$ ($-c=c_1$)

$\log xv - v = c_1$

$\log x(\frac{y}{x}) - \frac{y}{x} = c_1$

$\log y - \frac{y}{x} = c_1$

Taking e power

$e^{(\log y - \frac{y}{x})} = e^{c_1}$

$e^{\log y}e^{\frac{-y}{x}} = c_1$

$ye^{\frac{-y}{x}} = c_1$

Example 2:

Solve $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

Solution:

Given $(y^2 - 2xy)dx = (x^2 - 2xy)dy$

$\frac{dy}{dx} = \frac{(y^2-2xy)}{(x^2-2xy)}$ ………….. (1)

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

(1)⇒ $v + x\,\frac{dv}{dx} = \frac{v^2x^2-2vx^2}{x^2-2x^2v}$

$$v + x\,\frac{dv}{dx} = \frac{v^2x^2 - 2vx^2}{x^2 - 2x^2v}$$

$$v + x\,\frac{dv}{dx} = \frac{x^2(v^2 - 2v)}{x^2 - 2x^2v}$$

$$\Rightarrow x\frac{dv}{dx} = \left(\frac{v - 2v}{1 - 2v}\right) - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - 2v - v + 2v^2}{1 - 2v}$$

$$x\frac{dv}{dx} = \frac{3v^2 - 3v}{1 - 2v}$$

$$\Rightarrow \frac{dv(1 - 2v)}{3(v^2 - v)} = \frac{dx}{x} \;\ldots\ldots\ldots.(2)$$

Take $t = v^2 - v$

$$dt = (2v - 1)dv$$

$$-dt = (1 - 2v)dv$$

$$(2) \Rightarrow \frac{-dt}{3t} = \frac{dx}{x}$$

Integrating on both sides,

$$\int \frac{-dt}{3t} = \int \frac{dx}{x}$$

$$-\frac{1}{3}\log t = \log x + \log c$$

$$\log x + \frac{1}{3}\log t = \log c$$

$$\log x + \frac{1}{3}\log(v^2 - v) = \log c$$

$$\log\left(x(v^2 - v)^{1/3}\right) = \log c$$

Taking e power in both sides,

$$x(v^2 - v)^{1/3} = c$$

$$x\left(\frac{y^2}{x^2} - \frac{y}{x}\right)^{1/3} = c$$

$$\left(\frac{y^2}{x^2} - \frac{y}{x}\right)^{1/3} = c/x$$

$$\frac{y^2}{x^2} - \frac{y}{x} = \frac{C}{x^3}$$

$$\frac{y^2 - xy}{x^2} = \frac{C}{x^3}$$

$$x(y^2 - xy) = c$$

Example 3:

Solve $(x^2 + y^2)\, dy/dx \ = xy$

Solution:

Given $(x^2 + y^2)\frac{dy}{dx} = xy$

∴ This is the homogenous equation of degree 2

$$\frac{dy}{dx} = \frac{xy}{(x^2 + y^2)}$$

$$\frac{dy}{dx} = \frac{x^2(y \mid x)}{x^2(1 + y^2 \mid x^2)}$$

Put, $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{vx/x}{1 + v^2x^2/x^2}$$

$$x\frac{dv}{dx} = \frac{v}{1 + v^2} - v$$

$$x\frac{dv}{dx} = \frac{v - v^3}{1 + v^2}$$

$$x\frac{dv}{dx} = -v^3/1 + v^2$$

$$dv\frac{1 + v^2}{v^3} = -\frac{dx}{x}$$

$$0(1/v^3 + v^2/v^3) = -dx/x$$

$$(1/v^3 + 1/v) = -dx/x$$

Integrating on both sides,

$$\Rightarrow \int v^{-3} dv + \int 1/v dv = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{v^{-2}}{-2} + \log v = -\log x + \log c$$

$$\Rightarrow -1/2v^2 + \log v + \log x = \log c$$

$$-1/2v^2 + \log vx = \log c$$

$$-1/2y^2/x^2 + \log y/xxx = \log c$$

$$-x^2 \log^2 + \log y = \log c$$

Taking e powen on b.s,

$$e^{-x^2/2y^2} e^{\log y} = c$$

$$ye^{-x^2/2y^2} = c$$

$$y = ce^{x^2/2y^2}$$

Example 4:

Solve $(x + y)^2 dx = 2x^2 dy$

Solution:

Given $(x + y)^2 dx = 2x^3 dy$

$\frac{dy}{dx} = \frac{(x+y)^2}{2x^2}$ ………… (1)

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore (1) \Rightarrow v + x\frac{dv}{dx} = \frac{(x + vx)^2}{2x^2}$$

$$v + x\frac{dv}{dx} = \frac{x^2(1 + v)^2}{2x^2}$$

$$v + x\frac{dv}{dx} = \frac{(1 + v)^2}{2}$$

$$x\frac{dv}{dx} = \frac{1 + v^2 + 2v}{2} - v$$

$$x\frac{dv}{dx} = \frac{1 + v^2 + 2v - 2v}{2}$$

$$x\frac{dv}{dx} = \frac{1 + v^2}{2}$$

$$2\text{dv}/1 + v^2 = \frac{dx}{x}$$

Integrating on both sides,

$$2\int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$2\tan^{-1}(v) = \log x + c$

$\log x - 2\tan^{-1}(v) + c = 0$

$\log x - 2\tan^{-1}(y \mid x) + c_1 = 0$

$\log x - 2\tan^{-1}(y/x) = c_1$

1.3. Type C: Non-homogenous equations of first degree in x and y:

Consider $(ax + by + c)dy/dx = Ax + By + C$ ………… (1)

Where a, b, c, A, B, C ane constants

Put $x = x + h$ & $y = y + k$

$dx = dx$ & $dy = dy$

$$\therefore (1) \Rightarrow (ax + ah + by + bk + c)\frac{dy}{dx} = Ax + Ah + By + Bk + C$$

$$\Rightarrow (ax + by + ah + bk + c)\frac{dy}{dx} = Ax + By + Ah + BK + C \;\; \ldots\ldots\ldots\ldots. (2)$$

If h, k be chosen to satisfy

$ah + bk + c = 0$ ………….. (3)

$Ah + Bk + c = 0$ ………….. (4)

$(2) \Rightarrow (ax + by)\frac{dy}{dx} = Ax + By$ …………(5)

This is homogenous in x and y can be solved by putting

$y = vx$

Note 1:

The above solution succeeds only if $h\&k$ can be found from (3) & (4)

(i.e) if $a/A \neq b/B$

Note 2:

If $\frac{a}{A} = \frac{b}{B}$ and $\frac{c}{C}$ be different from reach of these fractions, h & k cannot be obtained from (5) & (4)

$\therefore$ Hence the following method is adopted Put $\frac{a}{A} = \frac{b}{B} = \frac{\lambda}{\mathrm{m}}$

$ma = A; bm = B$

$\therefore (1) \Rightarrow (ax + by + c)dy/dx = (max + mby + \mathbb{C})$

Put $ax + by = v$

$$a + b\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow b\frac{dy}{dx} = \frac{dv}{dx} - a \Rightarrow \frac{dy}{dx} = \frac{dv/dx - a}{b}$$

$\therefore$ (6) $\Rightarrow (v + c)dy/dx = m(v) + \mathbb{C}$

$$dy/dx = \frac{mv + c}{v + c}$$

$$\frac{b\frac{dv}{dx} - a}{b} = \frac{mv + c}{v + c}$$

$$\Rightarrow \frac{dv}{dx} - a = b\left(\frac{mv + c}{v + c}\right)$$

$$\frac{dv}{dx} = a + b\left(\frac{mv + c}{v + c}\right)$$

The variables separated.

Hence the solution is

$$\frac{v}{a + bm} + \frac{b(mc - c)}{(a + bm)^2}\log(v(a + bm) + ac + bc) = x + c_1$$

Note 3:

If $\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$ · the equation is $\frac{dy}{dx} = m$

$\Rightarrow y = mx + c_1$

Example 1:

Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$

Solution:

Given $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ …………(1)

Put

$x = x + h \quad ; y = y + k$
$dx = dx \quad ; dy = dy$

$\therefore$ (1) $\Rightarrow \frac{dy}{dx} = \frac{x + h + 2y + 2k - 3}{2x + 2h + y + k - 3}$

Consider $h + 2k - 3 = 0$ ………..(3)

$2h + k - 3 = 0$ …………. (4)

Equation (3) & (4) solve

$k = 1$ in (3)
$\because h = 1$

$x = x + 1 \Rightarrow x = x - 1$
$y = y + 1 \Rightarrow y = y - 1$

This is a homogenous equation

$y = vx$

$\therefore$ Put$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ (5) $\Rightarrow v + x\frac{dv}{dx} = \frac{x+2vx}{2x+vx}$

$v + x\frac{dv}{dx} = \frac{x(1+2v)}{x(2+v)}$

$$x\frac{dv}{dx} = \left(\frac{1+2V}{2+V}\right) - V$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+2v-2v-v^2}{2+v}$$

$$x\frac{dv}{dx} = \frac{1-v^2}{2+v}$$

$$\Rightarrow \left(\frac{3}{2}\frac{1}{1-v} + \frac{1}{2}\frac{1}{1+v}\right)dv = \frac{dx}{x}$$

Integrating on both sides,

$$-\frac{3}{2}\log(1-v) + \frac{1}{2}\log(1+v) = \log x + \log c$$

$$\log(1-v)^{-3/2} + \log(1+v)^{1/2} = \log cx$$

$$\log(1-v)^{-3/2} - (1+v)^{1/2} = \log cx$$

Taking e power on both sides,

$$\Rightarrow (1-v)^{-3/2}(1+v)^{1/2} = cx$$

squaring on both sides

$$(1-v)^{-3}(1+v) = c^2x^2$$

$$(1+v) = c^2x^2(1+v)^3$$

$$(1+y/x) = c^2x^2(1-y/x)^3$$

$$\frac{x+y}{x} = c^2x^2\frac{(x-y)^3}{x^3}$$

$$(x+y) = c^2(x-y)^3$$

$$(x-1+y-1) = c^2(x-1-(y-1))^3$$

$$(x+y-2) = c^2(x-1-y+1)^3$$

$$(x+y-2) = c^2(x-y)^3$$

Example 2:

Solve $(2x-4y+3)\frac{dy}{dx} + (x-2y+1) = 0$

Solution:

Given, $(2x-4y+3)dy/dx + (x-2y+1) = 0$

Put, $v = x - 2y$

$\frac{dy}{dx} = \frac{1}{2}(1 - \frac{dv}{dx})$

$(1) \Rightarrow (2v+3)(1/2 - 1/2dv/dx) + (v+1) = 0$

$$\frac{2v}{2} - \frac{2vdv}{2dx}\frac{3}{2} - \frac{3}{2}\frac{dv}{dx} + v + 1 = 0$$

$$v - \frac{vdv \cdot 2}{2dx} + \frac{3}{2} - \frac{3}{2}\frac{dv}{dx} + v + 1 = 0$$

$$(2v + 5/2) - 1/2dv/dx(2v+3) = 0$$

$$\frac{4v+5}{2} - \frac{1}{2}\frac{dv}{dx}(2v+3) = 0$$

$$4v + 5 - dv/dx(2v+3) = 0$$

$$(4v+5)dx - (2v+3)dv = 0$$

$$\mathrm{dx} - \frac{(2v+3)}{4v+5}dv = 0 \ldots\ldots\ldots\ldots \ (2)$$

Put, $t = 4v + 5 \Rightarrow v = \frac{t-5}{4}$

$dt = 4dv \Rightarrow dv = \frac{dt}{4}$

$(2) \Rightarrow dx - \left(\frac{2(t-5)}{4} + 3\right)\frac{dt}{t} = 0$

$$dx - \left(\frac{t-5+6}{8t}\right)dt = 0$$

$$dx - \left(\frac{t+1}{8t}\right)dt = 0$$

$$dx(-\frac{t}{8t} - \frac{1}{8t})dt = 0$$

Integrating on both sides,

$$x - \frac{1}{8}[dt + \frac{dt}{t}] = 0$$

$$x - \frac{1}{8}[t + \log t] = 0$$

$$x - \frac{1}{8}[(4v+5) + \log(4v+5)] = c$$

$$\Rightarrow x - \frac{1}{8}(4(x-2y)+5) - \frac{1}{8}\log(4(x-2y)+5) = c$$

$$\Rightarrow x + c = \frac{1}{8}((4x-8y)+5) + \frac{1}{8}\log(4x-8y+5)$$

$$\Rightarrow x + c = \frac{(4x-8y)}{8} + \frac{5}{8} + \frac{1}{8}\log(4x-8y+5)$$

$$\frac{8x+4x+8y}{8}+c-\frac{15}{8}=\frac{1}{8}\log(4x-8y+5)$$

$$\frac{4x+8y}{8}+c_1=\frac{1}{8}\log(4(x-2y)+5))$$

$$(4x+8y)+c_1=\log(4(x-2y)+5)$$

Example 3:

Solve $\frac{dy}{dx}+\frac{10x+8y-12}{7x+5y-9}=0$

Solution:

Given $\frac{dy}{dx}+\frac{10x+8y-12}{7x+5y-9}=0 \ldots\ldots\ldots (1)$

put $\left.\begin{matrix} x & = x+h \Rightarrow dx = dx \\ y & = y+k \Rightarrow dy = dy \end{matrix}\right\} \rightarrow (A)$

$$(1) \Rightarrow \frac{dy}{dx}+\frac{10x+10h+8y+8k-12}{7x+7h+5y+5k-9}=0$$

$$\frac{dy}{dx}+\frac{10x+8y+(10h+8k-12)}{7x+5y+(7h+5k-9)}=0$$

$$\frac{dy}{dx}=-\frac{(10x+8y)}{7x+5y} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Where $\begin{matrix} 10h+8k-12=0 \\ 7h+5k-9=0 \end{matrix}$

$$\frac{h}{-72+60}=\frac{k}{-84+90}=\frac{1}{50-56}$$

$$\frac{h}{-12}=\frac{k}{6}=\frac{1}{-6}$$

$$h=-\frac{12}{-6}=2$$

$$k=\frac{6}{-6}=-1$$

by (A)

$\left.\begin{matrix} x=x+2 \Rightarrow x=x-2 \\ y=y-1 \Rightarrow y=y+1 \end{matrix}\right\} \Rightarrow$ (B)

Put $y=vx$

$$\frac{dv}{dx}=v+x\frac{dv}{dx}$$

$$(2) \Rightarrow v + x dv/dx = \frac{-10x - 8vx}{7x + 5vx}$$

$$v + x\frac{dv}{dx} = \frac{-10 - 8v}{7 + 5v}$$

$$x\frac{dv}{dx} = \frac{-10 - 8v - 7v - 5v^2}{7 + 5v}$$

$$x\frac{dv}{dx} = \frac{-5v^2 - 15v - 10}{7 + 5v}$$

$$x\frac{dv}{dx} = \frac{-5(v^2 + 3v + 2)}{7 + 5v}$$

$$\frac{(7 + 5v)}{v^2 + 3v + 2} dv = -5\frac{dx}{x}$$

$$\left(\frac{2}{v + 1} + \frac{3}{v + 2}\right) dv = -5\frac{dx}{x}$$

Integrating on both sides,

$$2\log(v + 1) + 3\log(v + 2) = -5\log x + \log c$$

$$\log(v + 1)^2(v + 2)^3 = \log cx^{-5}$$

Taking e power on both sides,

$$(v + 1)^2(v + 2)^3 = c\, x^{-5}$$

$$(\frac{y}{x} + 1)(\frac{y}{x} + 2) = \frac{c}{x^5}$$

$$\frac{(y + x)^2}{x^2} \frac{(y + 2x)^3}{x^3} = \frac{c}{x^5}$$

By (B);

$$(y + 1 + x - 2)^2(y + 1 + 2x - 4)^3 = c$$

$$(y + x + 1)^2(y + 2x - 3)^3 = c$$

Example 4:

Solve $(x + y - 1)dy = (x + y + 1)dx$

Solution:

Given $\frac{dy}{dx} = \frac{(x+y+1)}{(x+y-1)}$ …………….(1)

Let $x + y = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$(1) \Rightarrow \frac{dv}{dx} - 1 = \frac{v+1}{v-1}$$

$$\frac{dv}{dx} = \frac{v+1}{v-1} + 1$$

$$\frac{dv}{dx} = \frac{v+1+v-1}{v-1}$$

$$\frac{dv}{dx} = \frac{2v}{v-1}$$

$$\left(\frac{v-1}{v}\right) dv = 2dx$$

$$dv - \frac{1}{v} dv = 2dx$$

Integrating on both sides,

$$v - \log v = 2x + c$$
$$(x+y) - \log(x+y) = 2x + c$$
$$c = 2x - x - y + \log(x+y)$$
$$c = x - y + \log(x+y)$$

1.4.Type D: Linear Equation:

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the first degree.

The linear equation is of the form $\frac{dv}{dx}$ +Py=Q ……….(1)

where P and Q are functions of x only

consider $\frac{dy}{dx} + Py = 0$

(i.e.,) $\frac{dy}{y} + Pdx = 0$

The solution is $ye^{\int Pdx} = C$

Diff $e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = 0$

$e^{\int Pdx}$ is the integrating factor

(1)multiply $e^{\int Pdx} \Rightarrow (\frac{dy}{dx} + Py)e^{\int Pdx} = Qe^{\int Pdx}$

Integrating on both sides,

$$ye^{\int Pdx} = \int Qe^{\int Pdx} dx + C$$

This is the solution of (1)

Example 1:

Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

Solution:

Given $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

$$\frac{dy}{dx} + Py = Q$$

This equation is of linear eqn.

$$\therefore P = \cos x \; Q = \frac{1}{2}\sin 2x$$

consider

$$\int Pdx = \int \cos x dx = \sin x$$

$$e^{\int Pdx} = e^{\sin x}$$

The solution is $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$

$$ye^{\sin x} = \int \frac{1}{2}\sin 2x \, e^{\sin x} dx + c$$

$$= \int \frac{1}{2}\sin x \cos x e^{\sin x} dx + c$$

$$= \int ze^z dz + c$$

$$= ze^z - \int e^z dz + c$$

$$= ze^z - e^z + c$$

$$= \sin x e^{\sin x} - e^{\sin x} + c$$

$$ye^{\sin x} = e^{\sin x}(\sin x - 1) + c$$

Example 2:

Solve $x\frac{dy}{dx} + y\log x = e^x x^{1-1/2\log x}$

Solution:

Given $x\frac{dy}{dx} + y\log x = e^x x^{1-\frac{1}{2}\log x}$

$$\div x \Rightarrow \frac{dy}{dx} + \frac{\log x}{x} y = \frac{e^x x^{1-\frac{1}{2}\log x}}{x}$$

This is of linear equation

$$P = \frac{\log x}{x}; Q = e^x x^{-\frac{1}{2}\log x}$$

$$\int Pdx = \int \frac{\log x}{x} dx$$

$$= \int u \cdot du$$

$$= u^2/2$$

$$\int Pdx = \frac{(\log x)^2}{2}$$

$$e^{\int Pdx} = e^{\frac{(\log x)^2}{2}}$$

$$= e^{(\log x)^{\frac{\log x}{2}}}$$

$$e^{\int Pdx} = x^{\frac{\text{tog}\, x}{2}}$$

$$u = \log x$$

$$\int Pdx = \frac{(\log x)^2}{2} \frac{(\log x)^2}{2} = \log$$

$$du = 11xdx$$

$\therefore$ The soln is $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$

$$yx^{\frac{\log x}{2}} = \int e^x x^{-1/2\log x} x^{\frac{1}{2}\log x} dx + c$$

$$= \int e^x dx + c$$

$$yx^{\frac{\log x}{2}} = e^x + c$$

Example 3:

Solve: $(1-x^2)$ dy $1dx + 2xy = x\sqrt{1-x^2}$ given that $y = 0$ when $x = 0$

Solution:

Given $(1-x^2)dy/dx + 2xy = x\sqrt{1-x^2}$

$\div (1-x^2)$

$$\frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-x^2}}$$

$$dy/dx + \frac{2x}{1-x^2}y = \frac{x}{\sqrt{1-x^2}}$$

This is of linear equation

$$\therefore P = \frac{2x}{1-x^2} : Q = \frac{x}{\sqrt{1-x^2}}$$

$$\int Pdx = \int \frac{2x}{1-x^2}dx$$

$$= \int -\frac{dt}{t}$$

$$= -\log t$$

$$= -\log(1-x^2)$$

$$e^{JPdx} = e^{-\log(1-x^2)}$$

$$= (1-x^2)^{-1} = 1/1-x^2$$

$$t = 1-x^2$$

$$dt = -2xdx$$

$$-dt = 2xdx$$

The solution is $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$

$$\frac{y}{1-x^2} = \int \frac{x}{(1-x^2)^{1/2}} \cdot \frac{1}{(1-x^2)}dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}dx + c$$

$$= \int \frac{\sin\theta}{(1-\sin^2\theta)^{3/2}}\cos\theta d\theta + c$$

$$= \int \frac{\sin\theta}{(\cos^2\theta)^{3/2}}\cos\theta d\theta + c$$

$$= \int \frac{\sin\theta}{\cos^3\theta}\cos\theta d\theta + c$$

$$= \int \frac{\sin\theta}{\cos^2\theta} d\theta + C$$

$$= \int \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} d\theta + c$$

$$= \int \tan\theta\sec\theta d\theta + c$$

$$= \sec\theta + C$$

$$\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + C \quad \dots\dots\dots\dots(1)$$

$y = 0$ when $x = 0$

$0 = 1 + C$

$c = -1$

$$\therefore \theta \Rightarrow \frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - 1$$

$$= \frac{1}{\sqrt{1-\sin^2\theta}}$$

$\sec\theta = 1/\sqrt{1-x^2}$

Example 4:

Solve $\frac{dy}{dx} + y\tan x = \cos^3 x$

Solution:

Given $\frac{dy}{dx} + y\tan x = \cos^3 x$

This is the linear eqn in xy

$\therefore P = \tan x \quad Q = \cos^3 x$

consider $\int Pdx = \int \tan x dx = -\log \cos x$

$$= \log(\cos x)^{-1}$$

$$e^{\int Pdx} = e^{\log(\cos x)^{-1}} = \frac{1}{\cos x}$$

∴ The solution is $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + C$

$$\frac{y}{\cos x} = \int \cos^3 x \cdot \frac{1}{\cos x} dx + c$$

$$= \int \cos^2 x dx + c$$

$$= \int \left(\frac{1 + \cos 2x}{2}\right) dx + c$$

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin 2x}{2} + c$$

$$\frac{y}{\cos x} = \frac{x}{2} + \frac{1}{4}\sin 2x + c$$

$$4y = \cos(2x + \sin 2x) + c$$

Example 5:

Solve $\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2}$

Solution:

Given $\frac{dy}{dx} - \frac{xy}{1-x^2} = \frac{1}{1-x^2}$ ……….(1)

This is the linear equation in y

$P = \frac{-\mathrm{x}}{1-x^2}$, $Q = \frac{1}{1-x^2}$

$\int Pdx = \int \frac{-x}{1-x^2} dx$

$t = 1 - x^2$

$= \int dt/2t$

$dt = -2xdx$

$= \frac{1}{2}\log t$

$= \frac{1}{2}\log(1-x^2)$

$= \log(1-x^2)^{1/2}$

$\therefore e^{\int Pdx} = e^{\log(1-x^2)^{1/2}}$

$= (1-x^2)^{1/2}$

$ye^{JPdx} = \int Qe^{JPdx}dx + c$

$y(1-x^2)^{1/2} = \int \frac{1}{(1-x^2)}(1-x^2)^{1/2}dx + c$

$= \int \frac{dx}{\sqrt{1-x^2}\cdot\sqrt{1-x^2}} \cdot \sqrt{1-x^2}dx + c$

$= \int \frac{dx}{\sqrt{1-x^2}} + c$

$y(1-x^2)^{1/2} = \sin^{-1} x + c$

Example 6:

Solve $(1+x^2)dy1dx + y = e^{\tan^{-1} x}$

Solution:

Given$(1+x^2)dy/dx + y = e^{\tan^{-1} x}$

$\div (1+x^2)\frac{dy}{dx} + y/(1+x^2) = \frac{e^{\tan^{-1} x}}{1+x^2}$

This is the linear equation in (3) y

$$P = \frac{1}{(1+x^2)} \quad Q = \frac{e^{\tan^1 x}}{1+x^2}$$

$$\int Pdx = \int \frac{dx}{1+x^2} \Rightarrow \tan^{-1} x$$

$$e^{\int Pdx} = e^{\tan^{-1} x}$$

$$ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$

$$ye^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x}}{1+x^2} \cdot e^{\tan^{-1} x} \cdot dx + c$$

$$= \int \frac{e^{2\tan^{-1} x}}{1+x^2} dx + c$$

$$= \int e^{2t} \cdot dt + C$$

$$= \frac{e^{2t}}{2} + c$$

$$ye^{\tan^{-1} x} = \frac{e^{2\tan^{-1} x}}{2} + c$$

Example 7:

Solve $\frac{dy}{dx} + \frac{3x^2 y}{1+x^3} = \frac{\sin^2 x}{1+x^3}$

Solution:

$$\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{\sin^2 x}{1+x^3}$$

This is linear equation in xy

$$P = \frac{3x^2}{1+x^3}, Q = \frac{\sin^2 x}{1+x^3}$$

$$t = 1 + x^3$$

$$\int pdx = \int \frac{3x^2}{1+x^3} dx$$

$$dt = 3x^2 \cdot dx$$

$$= \int dt/t$$

$$= \log t \Rightarrow \log(1+x^3)$$

$$e^{\int Pdx} e^{\log(1+x^3)}$$
$$= (1 - x^3)$$
$$ye^{\int Pdx} = \int Qe^{\int Pdx} \cdot dx + C$$
$$y(1+x^3) = \int \frac{\sin x}{1+x^3} \cdot (1+x^3)dx + c$$
$$= \int \frac{1-\cos 2x}{2} \cdot dx + c$$
$$= 1/2x - \frac{15\text{in } 2x}{4} + c$$
$$= \frac{x}{2} - \frac{\sin x \cos x}{4} + c$$
$$= \frac{x}{2} - \frac{\sin x \cos x}{2} + c$$
$$2y(1+x^3) = x - \sin x \cos x + c$$

Example 7:

Solve $\frac{dy}{dx} - 2\frac{y}{x} = \frac{5x^3}{(2+x)(3-2x)}$

Solution:

Given $\dfrac{dy}{dx} - 2y/x = \dfrac{5x^3}{(2+x)(3-2x)}$

This is the linear equation,

$$P = -\frac{2}{x} \quad Q = \frac{5x^3}{(2+x)(3-2x)}$$
$$\int Pdx = \int -\frac{2}{x}dx \Rightarrow -2\log x = \log x^{-2}$$
$$e^{JPdx} = e^{\log x^{-2}}$$
$$= \frac{1}{x^2}$$
$$ye^{\int Pdx} = \int Qe^{\int Pdx} \cdot dx + C$$
$$\frac{y}{x^2} = \int \frac{5x^3}{(2+x)(3-2x)} \frac{dx}{x^2} + c$$
$$= 5\int \frac{xdx}{(2+x)(3-2x)} + c$$

$$\frac{x}{(2+x)(3-x2)} = \frac{A}{2+x} + \frac{B}{3-2x}$$

$$x = A(3-2x) + B(2+x)$$

$$x = -2$$

$$-2 = 7A$$

$$A = \frac{2}{7}$$

$$x = \frac{3}{2}$$

$$\frac{3}{2} = \frac{7}{2}B$$

$$3 = 7B$$

$$B = \frac{3}{7}$$

$$= 5\left[\int \frac{-2}{7(2+x)}dx + \int \frac{3}{7}\frac{dx}{(3-2x)}\right] + c$$

$$= \frac{5}{7}\left[-2\log(2+x) + \frac{3}{-2}\log(3-2x)\right] + c$$

$$\frac{y}{x^2} = \frac{5}{7}[-2\log(2+x) - \frac{3}{2}\log(3-2x)] + c$$

$$7y = 5x^2[-2\log(2+x) - \frac{3}{2}\log(3-2x)] + c$$

1.5. Type E:Bernouilli's Equation:

Consider $\frac{dy}{dx} + Py = Qy^n$ …………….(1)

Where P and Q are functions of x only.

This can be reduced to the linear equation

Put $z = y^{1-n}$

$$\frac{dz}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

∴ (2) reduces to

$$\frac{1}{1-n}\frac{dz}{dx} + Pz = Q$$
$$\frac{dz}{dx} + (1-n)zP = Q(1-n)$$

This being linear in z, can be integrated by the method of solving linear equation and hence y can be got.

Example 1:

Solve $\frac{dy}{dx} - y\tan x = \frac{\sin x\cos^2 x}{4^2}$

Solution:

Given $\frac{dy}{dx} - y\tan x = \frac{\sin x\cos^2 x}{y^2}$

(1)multiply y^2 $y^2 dy/dx - y^3\tan x = \sin x\cos^2 x$

Put $z = y^3$

$$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{1}{3}\frac{dz}{dx} = y^2\frac{dy}{dx}$$

∴ (2) reduces to

$$\frac{1}{3}\frac{dz}{dx} - \tan xz = \sin x\cos^2 x$$
$$\frac{dz}{dx} - 3\tan xz = 3\sin x\cos^2 x$$

This is linear equation in z

$$P = -3\tan x\,,\ Q = 3\sin x\cos^2 x$$
$$\int Pdx = \int -3\tan xdx$$
$$= -3\log(\sec x) + c$$
$$e^{\int Pdx} = \log(\sec x)^{-3}$$

$$e^{\int Pdx} = \sec^{-3} x \Rightarrow \frac{1}{\cos^{-3} x} = \cos^3 x$$

$$z e^{\int pdx} \cdot \int Qe^{\int Pdx} dx + C$$

$$2\cos^3 x = \int 3\sin x \cos^3 x \cos^2 x dx + c$$

$$2\cos^3 x = \int 3\sin x \cos^3 x \cos^2 x dx + c$$

$$z\cos^3 x = 3\int \sin x \cos 5x dx + c \; t = \cos x$$

$$z\cos^3 x = -3\int t^5 dt + c \;\mathrm{dt}\;\; d\sin x dx$$

$$z\cos^3 x = -\frac{3}{6}[t^6] + C$$

$$y\cos^3 x = -\frac{1}{2}\cos^6 x + c$$

Example 2:

Solve: $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$.

Solution:

$$\text{Given } (x+1)\frac{dy}{dx} + 1 = 2e^{-y}$$

$$multiply\ by\,, e^{-y}$$

$$e^y(x+1)\frac{dy}{dx} + e^y = e^y \quad \dots\dots\dots\dots\dots.(1)$$

Take

$$z = e^y$$

$$\frac{dz}{dx} = e^y \frac{dy}{dx}$$

$$\therefore (1) \Rightarrow (x+1)\frac{dz}{dx} + z = 2$$

$$\div (x+1)$$

$$\frac{dz}{dx} + \frac{z}{x+1} = \frac{2}{x+1}$$

This is the Linear Equation in z

Here,

$$P = \frac{1}{x+1}; \; Q = \frac{2}{x+1}$$

$$\int Pdx = \int \frac{dx}{x+1} = \log(x+1)$$

∴ The solution is

$$Ze^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$

$$Z(x+1) = \int \frac{2}{x+1}(x+1)dx + c$$

$$z(x+1) = 2x + c$$

$$\Rightarrow e^y(x+1) = 2x + c.$$

Example 3:

Solve: $\frac{dy}{dx} + y\cos x = y^n \sin 2x.$

Solution:

$$\frac{dy}{dx} + y\cos x = y^n \sin 2x.$$

$$\div y^n \Rightarrow \; y^{-n}\frac{dy}{dx} + y^{1-n}\cos x = \sin 2x \quad \text{............ (1)}$$

Take

$$z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{1-n-1}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

$$\therefore (1) \Rightarrow \frac{1}{1-n}\frac{dz}{dx} + \cos z = \sin 2x$$

$$Multiply\ (1-n) \Rightarrow \frac{dz}{dx} + (1-n)\cos xz = (1-n)\sin 2x$$

This is the linear equation in x.

Hese $P = (1-n)\cos x$

$$\therefore \int Pdx = (1-n)\int \cos x dx$$

$$= (1-n)\sin x$$

$$e^{\int Pdx} = e^{(1-n)\sin x}$$

The solution is

$$ze^{\int Pdx} = \int Qe^{\int Pdx}dx + C$$

$$ze^{(1-n)\sin x} = \int (1-n)\sin 2xe^{(1-n)\sin x}dx + c$$

$$= 2(1-n)\int \sin x\cos xe^{(c-n)\sin x}dx + c$$

$$= 2(1-n)\int te^{(1-n)t}dt + c$$

$$v = \frac{e^{((-n)t}}{(1-n)}$$

$$= 2(1-n)\left[\frac{te^{(1-n)t}}{1-n} - \int \frac{e^{(1-n)t}}{1-n}dt\right] + c$$

$$= 2\left[e^{(1-n)t} \cdot \frac{e^{(1-n)t}}{1-n}\right] + c$$

$$Ze^{(1-n)\sin x} = 2\left[\sin xe^{(1-n)\sin x} \cdot \frac{e^{(1-n)\sin x}}{i-n}\right] + c$$

$$y^{1-n}e^{1-n(1-n)\sin x} = 2\left[\sin xe^{(1-n)\sin x} - \frac{e^{(1-n)\sin x}}{1-n}\right] + c.$$

Example 4:

Solve $\frac{dy}{dx} - \frac{2}{x}y = \frac{y^3}{x^3}$

Solution:

$$\frac{dy}{dx} - \frac{2}{x}y = \frac{y^3}{x^3}$$

$\div y^3 \Rightarrow$

$y^{-3}\frac{dy}{dx} - \frac{2}{x}y^{-2} = \frac{1}{x^3}$ .... (1)

put $z = y^{-2}$

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx}$$

$$\Rightarrow \frac{-1}{2}\frac{dz}{dx} - \frac{2}{x}z = \frac{1}{x^3}$$

$\therefore$ (3) $\quad x(-2)$

$$\frac{dz}{dx} + \frac{4}{x}z = \frac{-2}{x^3}$$

This is the linear equation in Z.

Here, $P = \frac{4}{x^2}$, $Q = -2/x^3$

Consider,

$$\int Pdx = \int \frac{4}{x} dx = 4\log x = \log x^4$$

$$e^{\int Pdx} = x^4$$

The solution is

$$ze^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$

$$zx^4 = \int -\frac{2}{x^3} x^4 dx + c$$

$$zx^4 = -2\int xdx + c$$

$$= -2\frac{x^2}{2} + c$$

$$y^{-2}x^4 = -x^2 + c$$

$$y^{-2}x^4 + x^2 = c$$

$$x^2\left(\frac{x^2}{y^2} + 1\right) = c\; x^2(x^2 + y^2) = cy^2$$

1.6. Exact Differential Equation:

An exact differential equation is obtained by equating exact or perfect differential to zero. Now we investigate the condition that a given differential equation may is e exact and the method of Integrating is then when the condition is satisfied.

Condition 1:

Let $Mdx + Ndy = 0$ be the differential equation

If this is exact, $\mu dx + Ndy$ must have been obtained by derivating some function $u(x, y)$ and performing no other operation.

$\therefore du = Mdx + Ndy.$

But $du = \frac{\partial u}{\partial x} dx + \frac{\delta u}{\partial y} dy.$

Hence the necessary condition for the go eqn sh exact are

$$M = \frac{\delta u}{\partial x}\; N = \frac{\delta u}{\partial y}$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\delta^2 u}{\partial y\, \partial x} = \frac{\delta u}{\partial x\, \partial y} = \frac{\delta N}{\partial x}$$

Hence $\frac{\delta M}{\partial y} = \frac{\delta N}{\partial x}$ is the criterion for $Mdx + \mathrm{N}\, dy = 0$ to be exact

Condition 2:

This condition is also sufficient.

If there be a function $u(x, y)$ whose differential du=Mdx + Ndy, we shall integrates relatively to x.

As the partial differential Mdx could have been derived only from the term containing x,

$$u = \int Mdx + \text{ terms not containing x}$$
$$= \int Mdx + F(y) \dots\dots\dots\dots(1)$$

Differentiating with respect to y partially,

$$\frac{\partial u}{\partial y} = \int \frac{\partial Mdx}{\partial y} + F'(y)$$

As, $N = \frac{\delta u}{\partial y}$

$F'(y) = N - \int \frac{\partial M}{\partial y} dx$ ………….. (2)

Differentiating with respect to x,

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

Integrating with respect to y,

$$F(y) = \int N - \int \frac{\partial M}{\partial y} dx\, dy + C$$

where C is an arbituary constant-

(1) $\Rightarrow$ $0 = \int Mdx + \int \left(N - \int \frac{\partial M}{\partial y} dx\right) dy + c$ is the primitive required.

Condition 3:

Practical Rule for solving exact differential equation:

Integrate Mdx as if y were constant and those terms in N dy that do not give the terms already occurring. The sum of these integrals equated to a constant gives the solution

$\int Mdx$ (treating y as constant) + $\int$ terms in N not containing $x\ dy = c$

Example 1:

Solve $(a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$

Solution:

Given $(a^2 - 2xy - y^2)dx - (x + y)^2 dx = 0.$ ………..(1)

Here

$$M = a^2 - 2xy - y^2$$
$$N = -(x + y)^2$$

$$\frac{\partial M}{\partial y} = -2x - 2y$$
$$\frac{\partial N}{\partial x} = -2(x + y)$$

Here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ (1) is an exact differential equation

Consider $(a^2 - 2xy - y^2)dx$

Integrating with respect to x, we get

$$a^2x - \frac{2x^2}{2}y - y^2x$$

Differentiating with respect to y

$-x^2 - 2yx = -(x^2 + 2xy)$

The only, term left over in $-(x + y)^2dy$ is y^2dy

∴ The solution is $a^2x - x^2y - y^2x - \frac{y^3}{3} = c$

Example 2:

Solve $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)dx = 0$

Solution:

$$(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)dx = 0$$

$$M = 12x^3y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y$$
$$N = 2x^2y + 4x^3 - 12xy^2 + 3y^2 = xe^y + e^{2x}$$

Now,

$$\frac{\partial M}{\partial y} = 12x^2 + 4xy + 12x^2 - 12y^2 + 2e^{2x} - e^y$$
$$\frac{\partial N}{\partial x} = 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}$$

here $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

∴ The given equation is exact

∴ The solution.

$\int Mdx + \int$ N terms in N not containing $xdy = C$.

$$\int 12x^2y + 2xy^2 + 4x^3 - 4y^3 - 4y^3 + 2ye^{2x} - e^y\Big) dx + \int 3y^2 dy \cdot c$$
$$\Rightarrow \frac{12x^3y}{3} + \frac{2x^2y^2}{2} + \frac{4x^4}{4} - 4xy^3 + \frac{2ye^{2z}}{2} - e^y x + \frac{3y^3}{3} = 0$$
$$\Rightarrow 4x^3y + x^2y^2 + x^4 - 4xy^3 + ye^{2x} - xe^y + y^3 = c$$

Note:

Sometimes an equation can be rendered exact by multiplying the equation by a suitable integrating factor (I.F).

Rules for finding Integrating factors:

1. When $Mx + Ny \neq 0$ and the equation is homogeneous. $\frac{1}{Mx+Ny}$ is an IF of $Mdx + Ndy = 0$.
2. When $Mx = Ny + 0$ and the equation of is the form $f_1(xy)\, ydx + f_2(xy)\, ydx = 0$ $\frac{1}{Mx-Ny}$ is an IF
3. If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say $f(x)$ then the IF is $e^{\int f(x)dx}$.
4. If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ a in of y alone say $f(y)$ then the I. gie $e^{\int f(y)dy}$.

Example 1:

Solve: $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

Solution:

$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ ……………(1)

Here

$$M = x^2y - 2xy^2$$
$$N = -(x^3 - 3x^2y)$$
$$\frac{\partial M}{\partial y} = x^2 - 4xy$$
$$\frac{\partial N}{\partial x} = -(3x^2 - 6xy)$$
$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial yx}$$

∴ The given equation is not exact

since the given equation is homo 1.

$$\therefore I\cdot F\,\frac{1}{Mx+Ny}$$

$$I=\frac{1}{x^3y-2x^2y^2-x^3y+3x^2y^2}$$

$$I\cdot F=\frac{1}{x^2y^2}$$

$$\frac{x^2y-2xy^2}{x^2y^2}dx-\frac{(x^3-3x^2y)dy}{x^2y^2}=0$$

$$\left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0 \quad \ldots\ldots\ldots\ldots\ldots..(2)$$

Here

$$M_1=\frac{1}{y}-\frac{2}{x}$$

$$N_1=-\left(\frac{x}{y^2}-\frac{3}{y}\right)$$

$$\frac{\partial M_1}{\partial y}=\frac{-1}{y^2}\cdot\frac{\partial N_1}{\partial x}=-\frac{1}{y^2}$$

$$\therefore\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$$

∴ The equation is exact.

∴ The solution is

$\int M_1dx+\int$ terms in N_1 Not confaining $xdy=c$.

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$$\Rightarrow\frac{x}{y}-2\log x+3\log y=c$$

$$\Rightarrow\frac{x}{y}+\log x^2+\log y^3=c$$

$$\Rightarrow\frac{x}{y}+\log x^{-2}y^3=c$$

$$\Rightarrow\frac{x}{y}+\log\frac{y^3}{x^2}=c$$

Example 2:

Solve: $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$.

Solution:

The equation is of the form $f_1(xy)ydx+f_2(xy)xdy=0$.

By rule 2, the I.F. $=\frac{1}{Mx-Ny}$

$$=\frac{1}{xy(xy+2x^2y^2)-xy(xy-x^2y^2)}$$

$$=\frac{1}{3x^3y^3}$$

Multiplying the equation by $\frac{1}{3x^3y^3}$, we have

$$\frac{y(xy+2x^2y^2)}{3x^3y^3}dx + \frac{x(xy-x^2y^2)}{3x^3y^3}dy = 0$$

$$\text{i.e., } \left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy = 0$$

This is exact. Hence $\int \left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx$ (treating y as constant) $+\int -\frac{1}{3y}dy$ (only these terms not containing x) $=$ C.

$$\therefore \log\frac{x^2}{y} - \frac{1}{xy} = C$$

Example 3:

Solve $(y-3x^2)dx - x(1-xy^2)dy = 0$.

Solution:

The equation can be written in the form

$$ydx - xdy - 3x^2dx + x^2y^2dy = 0$$

Dividing by x^2, we get $\frac{ydx-xdy}{x^2} - 3dx + y^2dy = 0$

$-d\left(\frac{y}{x}\right) - d(3x) + \frac{1}{3}d(y^3) = 0. \therefore -\frac{y}{x} - 3x + \frac{1}{3}y^3 =$ C.

Example 4:

Solve $\frac{dy}{dx} = \frac{2x}{x^2+y^2-2y}$.

Solution:

The equation can be written in the form

$$(x^2+y^2)dy = 2ydy + 2xdx$$

$$dy = \frac{d(y^2+x^2)}{x^2+y^2}$$

$dy = d\log(x^2+y^2)$. $\therefore\ y = \log(x^2+y^2) + \text{C}$

Example 5:

Solve $(1+xy^2)dx + (1+x^2y)dy = 0$.

Solution:

The equation can be written in the form

$$dx + dy + xy(ydx + xdy) = 0$$

$$\text{i.e., } d(x+y) + xyd(xy) = 0$$

$$\text{i.e., } d(x+y) + \frac{1}{2}d(xy)^2 = 0$$

$$\therefore x + y + \frac{1}{2}(xy)^2 = \text{C}.$$

Example 6:

Solve $(x^2+y^2)(xdx+ydy) = a^2(xdy - ydx)$

Solution:

This equation can be written in the form

$$xdx + ydy = a^2\frac{xdy - ydx}{x^2+y^2}$$

$$\frac{1}{2}d(x^2+y^2) = a^2 d\tan^{-1}\left(\frac{y}{x}\right)$$

$$\therefore \frac{1}{2}(x^2+y^2) = a^2\tan^{-1}\left(\frac{y}{x}\right) + \text{C}$$

Exercises:

1. Solve $(x^3y^3 + x^2y^2 + xy + 1)ydx + (x^3y^3 - x^2y^2 - xy + 1)xdy) = 0$
2. Solve $ydx - xdy - 3x^2y^2e^{x^3}dx = 0$
3. Solve $(x^2 - yx^2)\frac{dy}{dx} + (y^2 + x^2y^2) = 0$.
4. Solve $(x^2 - x + y^2)\text{dx} + (ye^{\text{y}} - 2\text{xy})\text{dy} = 0$.

UNIT II:

Equation of first order but of higher degree: Equation solvable for dy/dx - Equation solvable for y – Equation solvable for x – Clairaut's form – Linear Equations with constant coefficients: Definition – The operator D – Complete solution – Particular integrals of algebraic, exponential, trigonometric functions and their products.

Chapter 2: Sections – 2.1-2.7

2.Equation of first Order but not of Higher degree:

2.1.Type A:Equations solvable for $\frac{dy}{dx}$

we shall denote $\frac{dy}{dx}$ hereafter by P. Let the equation of the first order and of the n^{th} degree be

$P^n +| P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0$

where $P_1, P_2, \dots, P_n$ denote functions of x and y Suppose the first member of (1) can is resolved into factors of the first degree of the form

$(p - R_1)(p - R_2)(p - R_3) \cdots (p - R_n)$

Any relation between x & y which makes any of these functions factors vanish is a solution of (1). let the primitives of

$P - R_1 = 0, P - R_2 = 0$, etc. be

$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, a_n = (x, y, z_n) = 0$ respectively, where c_1, c_2 are arbitary constants without any loss of generality, we can replace $c_1, c_2, \dots c_n$ by c.

Where c is an arbitrary constant.

Hence, the solution of (1) is

$\phi_1(xyc) \cdot \phi(xyc) \cdots \phi_n(xyc) = 0.$

Example 1:

1.Solve: $x^2p^2 + 3xyp + 2y^2 = 0$

Solution:

Given $x^2p^2 + 3xyp + 2y^2 = 0$ ... ...(1)

$$\left(p + \frac{2y}{x}\right)(p + \frac{y}{x}) = 0$$

$$P + \frac{2y}{x} = 0;\ P + \frac{y}{x} = 0$$

Now consider,

$$p+\frac{2y}{x}=0$$

$$\frac{dy}{dx}=-\frac{2y}{x}$$

$$\frac{dy}{y}=-\frac{2dx}{x}$$

$$\log y=-2\log x$$

$$\log y=-\log x^2+\log c$$

$$\log y+\log x^2=\log c$$

Taking e power on both side

$x^2y=c$

Consider

$$P+\frac{y}{x}=0$$

$$\frac{dy}{dx}+\frac{4}{x}=0$$

$$\frac{dy}{dy}=\frac{-dx}{x}$$

$$\log y=\log x+\log c$$

$$\log y+\log x\log c$$

Taking e power on both sides

$xy=c$

From (2) & (3)

$(x^2y-c)(xy-c)=0$ is the required solution.

Example 2:

Solve: $p^2+\left(x+y-\frac{2y}{x}\right)p+\text{xy}+\frac{y^2}{x^2}-y-\frac{y^2}{\text{x}}=0$

Solution:

Given $p^2+\left(x+y-\frac{2y}{x}\right)p+xy+\frac{y^2}{x}=0$

$$\left(p+x-\frac{y}{x}\right)\left(p+y-\frac{y}{x}\right)=0$$

$$\left(p+x-\frac{y}{x}\right)=0$$

Consider,

$$P + x - \frac{y}{x} = 0$$
$$P - \frac{y}{x} = -x$$
$$\frac{dy}{dx} - \frac{y}{x} = -x$$

This is a linear equation.

$$P = \frac{-1}{x}, Q = -x.$$
$$\int pdx = -\log x \Rightarrow e^{\int Pdx} = e^{-\log x} = 1/x$$

The Solution is

$$ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$$
$$y/x = \int -\frac{x}{x} dx + c$$
$$y/x = -x + c$$

Consider:

$$p + y - y/x = 0$$
$$\frac{dy}{dx} + y\left(1 - \frac{y}{x}\right) = 0$$

$$\frac{dy}{y} = -dx + \frac{dx}{x}$$

Integrating on both sides,

$$\log y = -x + \log x + \log c$$
$$\log y + x - \log x = \log c$$
$$\log(y/x) + x = \log c$$

Taking e power on both sides

$$\frac{y}{x} \cdot e^x = c$$

From (2) & (3)

$(y + x^2 - cx)(y - cxe^{-x}) = 0$, is the required solution.

2.2.Type B:

Let the differential equation (1) in section 2.1 be put in the form $f(x, y, p) = 0$. When it cannot be resolved into rational linear factors as in section 2.1, it may be either solved for y (or) x.

1. Equations solvable for y:

$f(x,y,p) = 0$ can be put in the form $y = F(x,p)$ ………..(1)

Differentiating with respect to x,

$p = \phi(x,p,dp/dx)$

This, being an equation in the two variables p and s, can be integrated by any of the methods. Hence we obtain,

$\psi(x,p,c) = 0$ ……..(2)

Eliminating p between (1) and (2) the solution is got.

2. Equation solvable for x

$f(x,y,p) = 0$ be in this case put in the form.

$x = F(y,p)$. ………. (1)

Differentiating with respect to y,

$1/p = \phi(y,p,dp/dx)$

This equation being in two variable p and y, can be integrated by any of the method.

Hence $\psi(\varphi,p,C) = 0$ ………...(2)

Eliminating p between (1) & (2) the solutions of (1) is got.

Example 1:

Solve $xp^2 - 2yp + x = 0$

Solution:

Given, $xp^2 - 2yp + x = 0$

$$xp^2 + x = 2yp$$
$$2yp = xp^2 + x$$
$$y = \frac{x(p^2+1)}{2p} \quad \dots\dots\dots\dots..(1)$$

Differentiating with respect to x,

$$\frac{dy}{dx} = \frac{2P\left[x\left(2P\frac{dP}{dx}\right) + (P^2+1)\right] - [x(P^2+1)2dP/dx]}{4P^2}$$

$$P = \frac{2P\left[x\left(2P\frac{dP}{dx}\right) + (P^2+1)\right] - [x(P^2+1)2dP/dx]}{4P^2}$$

$$4p^3 = 4p^2x\frac{dp}{dx} + 2p(p^2+1) - 2x^2p^2\frac{dp}{dx} - 2x\frac{dp}{dx}$$

$$4p^3 = 2p^2x\frac{dp}{dx} + 2p(p^2+1) - 2xdp/dx.$$

$$4p^3 = 2x\frac{dp}{dx}(p^2-1) + 2p(p^2+1)$$

$\div 4p^3$

$$1 = \frac{x\frac{dp}{dz}(p^2-1)}{2p^3} + \frac{p^2+1}{2p^2}$$

$$1 - \frac{p^2+1}{2p^2} = \frac{x(p^2-1)}{2p^3}\frac{dp}{dx}$$

$$\frac{2p^2-p^2-1}{2p^2} = \frac{x(p^2-i)}{2p^3}\frac{dp}{dx}.$$

$$\frac{p^2-1}{2p^2} = \frac{x\cdot(p^2-1)}{2p^3}\frac{dp}{dx}$$

$$1 = \frac{x}{p}\frac{dp}{dx}$$

$$\frac{dp}{p} = \frac{dx}{x}$$

Integrating on both sides,

$\log p = \log x + \log c$

$p = xc$ ………………. (2)

Sub (2) in (1).

$$y = \frac{x(c^2x^2+1)}{2cx}$$

$2\text{cy} = c^2x^2 + 1$

Example 2:

Solve: $x = y^2 + \log p$.

Solution:

Given $x = y^2 + \log p$ ………… (1)

Differentiating with respect to y

$$\frac{dy}{dy} = 2y + \frac{1}{p}\frac{dp}{dy}$$

$$\frac{1}{p} = 2y + \frac{1}{p}\frac{dp}{dy}$$

$$1 = 2yp + \frac{dp}{dy}$$

$$\frac{dp}{dx} + 2yp = 1$$

This is linear in P.

$P_1 = 2y;\ Q_1 = 1$

I. F $e^{\int Pdx} = e^{\int 2ydy}$

Hence $Pe^y = \int e^{y^2} dy + c$ ………… (2)

The eliminant of p between (1) & (2) gives the solution.

2.3.Particular case for solvable for y:

1.Clairaut's form.

The equation known as Clairaut's form.

$y = px + f(p)$ … … …. . (1)

Differentiating with respect to x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)dp.$$

$$p = p + (x + f'(p))\frac{dp}{dx}$$

$$0 = (x + f'(p))\frac{dp}{dx}$$

$$\Rightarrow \frac{dp}{dx} = 0\,, x + f'(p) = 0$$

Now

$$\frac{dp}{dx} = 0$$

$\Rightarrow\ P = C, a$ constant.

∴ The solution is

$y = cx + f(c)$.

Note:

1. We have to replace p in clairauts equation by C.
2. The other factor $x + f'(p) = 0$ taken along with equation (1) give on elimination of P, a solution of (1). such a solution called, a singular solution

Example 1:

Solve : $y = (x - a)p - p^2$.

Solution:

$y = (x - a)p - p^2$

This is Clairauts equation,

∴ The general solution is

$y = (x - a)c - c^2$

Example 2:

Solve: $y = 2px + y^2p^3$.

Solution:

Putting X=2x and Y=y^2, the equation transforms into

Y=XP+P^3 , where $P = \frac{dY}{dX} = py$

This is Clairaut's equation; hence Y=Xc+c^3

The solution is y^2=2xc+c^3

2.We have an extended from of Clairaut's equation of the type

$y = xf(p) + \phi(p)$ ………… (1)

Differentiating with respect to x

$$p = f(p) + [x\, f'(p) + \phi'(p)]\frac{dp}{dx}$$

$$\frac{dx}{dp} + \frac{xf'(p)}{f(p) - p} = \frac{\phi'(p)}{p - f(p)}$$

This is linear in x and hence gives F (x, p, c) =0

The eliminant of p between this equation and (1) gives the solution of (1).

Example 1:

Solve: $y = px + x(1 + p^2)^{1/2}$.

Solution:

Given

$y = px + x(1 + p^2)^{1/2}$ ………….. (1)

$y = \left[P + (1 + P^2)^{1/2}\right]x$

This is of extended Clairaute form.

Differentiating with respect to x,

$$\frac{dy}{dx} = P + x\frac{dP}{dx} + x\frac{1}{2}(1+P^2)^{-1/2}2P\frac{dP}{dx} + (1+P^2)^{\frac{1}{2}}$$

$$\Rightarrow \mathrm{p} = p + x\frac{dp}{dx} + \frac{px}{\sqrt{1+p^2}}\frac{dp}{dx} + (1+p^2)^{1/2}$$

$$\Rightarrow 0 = (1+p^2)^{\frac{1}{2}} + \frac{dp}{dx}\left[x + \frac{px}{\sqrt{1+x^2}}\right]$$

$$= \sqrt{(1+p^2)} = \frac{dp}{dx}\mathrm{x}\left[1 + \frac{p}{\sqrt{1+p^2}}\right]$$

$$\Rightarrow -\sqrt{1+p^2} = \frac{dp}{dx}\mathrm{x}\left[\frac{\sqrt{1+p^2}+p}{\sqrt{1+p^2}}\right]$$

$$\Rightarrow -\frac{dx}{x} = dp\left[\frac{\sqrt{1+p^2}+p}{(1+p^2)}\right]$$

Integrating on both side,

$$\Rightarrow \log\left(p\sqrt{1+p^2} + 1 + p^2\right) = -\log x + \log c$$

$$\Rightarrow \log\left(p\sqrt{1+p^2} + (1+p^2)\right) + \log x = \log c$$

$$\Rightarrow \log\left[p\sqrt{1+p^2} + (1+p^2)x\right] = \log c$$

Taking e power on both sides $\left[P\sqrt{1+P^2} + (1+P^2)\right]x = c$ ………….(2)

The eliminant of P between (1) & (2) gives the solution,

given $y = 2px + y^2p^3$

Take

$$x = 2x;\ y = y^2$$
$$dx = 2dx;\ dy = 2ydy.$$
$$\frac{dy}{dx} = \frac{2ydy}{2dz} = y\frac{dy}{dx}$$
$$p = y\ \Rightarrow p = \frac{P}{y}$$
$$\Rightarrow p = \frac{p}{\sqrt{y}}[\because y = y^2]$$

$\therefore$ (1) $\Rightarrow$

$\Rightarrow \sqrt{y} = px + yp^3$

$$\sqrt{y} = \frac{p}{\sqrt{y}} + y\left(\frac{p}{\sqrt{y}}\right)^3$$

$$\sqrt{y} = \frac{p}{\sqrt{y}}x + y\frac{p^3}{y\sqrt{y}}$$

$\Rightarrow y = px + p^3$

This is of Clairaut's form.

The solution is

$y = cx + c^3$

$y^2 = 2cx + c^3$

2.4. Linear Equation with constant Coefficients:

Definitions:

A linear equation is one in which the dependent variable y and its derivatives of any Order occur only in the first degrees are not multiplied together, their coefficients being Constants or functions of the independent Variable x.

Consider the equation.

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny = X \; \ldots\ldots\ldots\ldots.(1)$$

Where $P_1, P_2, \ldots P_n$ and x are functions of X or constants. We will first show that the complete solution of (1) contains, as part the complete solution of (1) without the second member, (i.e.,) that of $\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{ny} = 0$ …………(2)

If $y = y_1$ be a solution of (2), we easily see by substitution that $y = c_1y_1$, where c_1 is arbitrary constant, is a solution of (2).

Similarly, if $y = y_2, \; y = \mathrm{y}_3 \ldots \mathrm{y} = y_n$ be integral of (2) then $y_1 = c_2y_2, \ldots y = C_ny_n$ are also integral of (2) $c_2, c_3, \ldots c_n$ being arbitrary constants.

Thus $y = c_1y_1 + c_2y_2 + \cdots c_ny_n \; is \; a$ solution of (2) as cant verified by substitution.

If $y_1, y_2 \ldots y_n$ are linearly independent the above value of y is the complete solution of (2) as it contains n arbitrary constants, n being equal to order of the equation (2).

Let $y = u$ be a particular solution of (1) (where u does not contain any arbitrary constant) then $y = y + u$, where $y = c_1y_1 + \cdots + c_ny_n$ is a solution of (2).

This represents the complete solution of (1) as it coitus arbitrary constants. The pact of y is called the Complementary Function of (1) & u is called the particular integral of (1). Thus the primitive of (1) is the sum of the complementary function and the particular integral.

2.5.The Operator D:

Let D denote the Operator $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2} \cdot dt$.

The symbol. D satisfies the commutative, associative, distributive laws of Algebra.

For, if m and n are positive integers.

$$(D^m + D^n)v = (D^n + D^m)u$$
$$D^m \cdot D^n u = D^n \cdot D^m u = D^{m+n} u$$

and $D(u + v) = D(\mathrm{v} + u)$.

We can define the operator with negative indices by common analogy with algebra.

If $Du = v$ then u lan be written as D^{-1}. The operator D^{-1} is the inverse operator, it $\frac{1}{D}$.

Then $v = Du = D \cdot D^{-1} v$

i.e., $D \cdot D^{-1} = 1$.

The Inverse operator D^{-1} is one such that when it operates on any function of x and subsequently the operation by D be performed, the function is left unaltered. The inverse operators also obey algebraical laws. D^{-1} represents an integration

the equation (1) of 1 can be symbolically written as.

$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = X$

If we set $f(D) = D^n + P_1 D^{n-1} + \cdots + P_n(1)$ is $f(D)y = X$

2.6. Complementary function of a linear equation with constant coefficients:

Let us consider the equation $d^n y + P^{d^{n-1}y} + \cdots + P_n y = X$ ……………..(1)

where $P_1, \ldots P_n$ are constants and X a function of x,

i.e., $f(D)y = X$ …………(2)

The complementary function of (2) is the general solution of

$f(D)y = 0$ ………… (3)

Let us take a trial solution $y = e^{mx}$ of (2) for came value of m, As $D^r e^{mx} = m^r e^{mx}$, then Substitution of e^{mx} for y in (2) gives. $f(m)e^{mx} = 0$

Hence $f(m) = m^n + P_1 m^{n-1} + P_n = 0$. ……………(4)

This equation is called the auxiliary equation.

Case: 1

Let the auxiliary equation (4) have n distinct roots, say $m_1, m_2, \ldots m_n$ -

The complete sols of (3) is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

where $C_1, C_2, \dots C_n$ are arbitrary constants.

Case: 2

Let two of the roots of equation (3), say m_1 and m_2 be equal

The past of the complementary function arising from the two roots is $(c_1 + c_2)e^{m_1 x}$ & as $c_1 + c_2$ can be replaced by a single constant, the no.. of constants in general soln of (2) is reduced by one to $n - 1$.

Hence we Proceed as follows.:

Let us put $m_2 = m_1 + \epsilon$ & allow ϵ to tend to zero. The C.F. arising from these roots is

$$c_1 e^{m_1 x} + c_2 e^{(m_2+\epsilon)x}$$
$$= e^{m_1 x}[c_1 + c_2 e^{\epsilon x}]$$
$$= e^{m_1 x}\left[c_1 + c_2\left(1 + \epsilon x + \frac{\epsilon^2 x^2}{2!} + \cdots\right)\right] \text{ by the exponential theorem.}$$

$= e^{m_1 x}[c_1 + c_2 + c_2 \in x]$, the other terms tending to zero as $\in \to 0$

we can choose e_2 sufficiently big as to make $c_2 \in$ finite as $\epsilon \to 0$ and C_1 large leith Opposite orign to $G_1 + C_3, A$ & $C_2 C - B$, the C.F. conresponding to the Hwo equal voots m, is $e^{m,r}(A + Bx)$.

More generally, if r roots of (3) are equal to m_1, say the corresponding r terms in CF will apparently coalesce into a single term. But, by a similar reasoning to that we have adopted for two equal roots, the r terms in the C.F can be replaced by.

$e^{m_1 x}(A_1 + A_2 x + \cdots + A_1 x^{r-1})$.

Case 3:

let the auxiliary equation have imaginary roots. Imaginary roots always occur in pairs. Thus if α+ iβ a root of (3), α -iβ is also a root; α , β being real.

The corresponding terms of the C.F., are

$$e_1{}^{(\alpha+i\beta)x} + c_2 e^{(x-i\beta)x}$$
$$= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right].$$
$$= e^{dx}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)] \text{ by Euler's formula}$$

$y = e^{\alpha x}(n\cos\beta x + \beta\sin\beta x),$

where A & B are arbitrary constants.

Example 1:

Solve: $(D^2 - 5D + 4)y = 0$

Solution:

Given $(D^2 - 5D + 4)y = 0$

$m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m_1 = 1,4$ roots are distinct.

$\therefore$ The solution is $y = C_1 e^x + c_2 e^{4x}$.

Example 2:

Solve: $(D^3 - 3D^2 + 4)y = 0$.

Solution:

$(D^3 - 3D^2 + 4)y = 0$

$m^3 - 3m^2 + 4 = 0$

$(m+1)(m^2 - 4m + 4) = 0$

$(m+1)(m-2)^2 = 0$

$m = -1,2,2$

$y = c_1 e^{-x} + e^{2x}(c_2 + c_3 x)$

Example 3:

Solve: $(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$.

Solution:

$(D^4 - 4D^3 + 8D^2 - 8D + 4)y = 0$

The A.E is

$m^4 - 4m^3 + 8m^2 - 8\text{m} + 4 = 0$

$(m^2 - 2m + 2)^2 = 0$

$(m^2 - 2m + 2)\,(m^2 - 2m + 2) = 0$

$m = (1 \pm i)$

$= e^x(A + Bx)(c\,cosx + Dsinx)$

2.7. Particular Integral:

To find the particular integral of $f(D)y = x$, where X is any function of x only, various special methods are devised which depend on the form of x.

The particular integral is written symbolically as $\frac{1}{f(D)} \times$. we have already defined the inverse operator $\frac{1}{f(D)}$; accordingly $\frac{1}{f(D)} \times$ is a that function of x, which when operated upon by $f(D)$, yields X.

1.General method of finding Particular Integral:

$\frac{1}{f(D)}$ can either be broken up into factors which can be taken in any order into partial functions. If the roots of the auxiliary equations $f(m) = 0$ be $m_1, m_2, \ldots m_n$. then.

$f(D) = (D - m_1) \ldots \ldots . . (D - m_n)$.

Hence $\frac{1}{f(D)} \times$ can be put either in the form.

$$\frac{1}{D - m_1}, \frac{1}{D - m_2} \cdots \frac{1}{D - m_n} \mathrm{X}$$

$a\left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right) X$, where $A_1, A_2, \ldots A_n$ are constants.

This is first analogous to decomposition into partial functions for rational algebraic functions.

In either case, the evaluation of the P.I depends on $\frac{1}{D-a} x$.

Let $z = \frac{1}{D-\alpha} \times$ ie., $\frac{dz}{dx} - \alpha z = x$

This is a linear equation of the first order hence

$$ze^{-ax} = \int Xe^{-ax} dx$$

No constant should be added as this is o P.I

$$z = e^{ax} \int xe^{-ax} dx$$

2.Special Method for finding P.I.

i) $x = e^{ax}$

case 1:

In $\frac{1}{f(D)} e^{\mathrm{a}x}$, replace D by a if $f(D) \neq 0$

case: 2

if $f(a) = 0$ then. P.I $- \frac{e^{ax}}{\varphi(x)(D-a)^r}$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{\psi(a)} \frac{x^r e^{ax}}{r!}$$

Example 1:

Solve $(D^2 + 5D + 6)y = e^x$

Solution:

Given $(D^2 + 5D + 6)y = e^x$

The auxiliary equation is

$$m^2 + 5m + 6 = 0$$
$$(m + 2)(m + 3) = 0$$
$$m = -2, -3$$

∴ The complementary function is

$y = c_1 e^{-2x} + c_2 e^{-3x}$ .... (1)

To find P.I

$$y = \frac{e^x}{D^2 + 5D + 6} = \frac{e^x}{1 + 5 + 6}$$
$$= \frac{e^x}{12}$$

∴ The general solution is

$$y = c \cdot F + P \cdot \tau$$
$$y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{12} e^x$$

Example 2:

Solve: $(D^2 - 2mD + m^2)y = e^{mx}$

Solution:

Given $(D^2 - 2mD + m^2) \cdot y = e^{mx}$

The Auxiliary equation is,

$$k^2 - 2mk - m^2 = 0$$
$$(k - m)^2 = 0$$
$$k = m, m$$

The $C \cdot F$ is $y = e^{mx}(c_1 + c_2 x)$ ....(1)

Now, P.I

$$= \frac{e^{mx}}{D^2 - 2mD + m^2}$$
$$= \frac{e^{mx}}{(D - m)^2}$$
$$= \frac{x^2}{2} e^{mx}$$

∴ The general solution is,

$$y = e^{mx}(c_1 + c_2 x) + \frac{x^2}{2} e^{mx}$$

Let X be of the form cos ax or sin ax where a is a constant.

$X = \cos ax \ (or) sinax$

Replace $D^2 by - a^2$, provided.

Case 1: If $\phi(-a^2) \neq 0$

$\frac{1}{\phi(D^2)}\sin ax = \phi(-a^2)\sin ax$ if $\phi(D^2)$ be a rational integral function of D^2

Operating on both sides by $\frac{1}{\phi(D^2)}$ sin$ax = \frac{\phi(-a^2)}{\phi(D^2)}\ sinax$

Hence the rule is:

Replace D^2 by $-a^2$, provided $\phi(-a^2) \neq 0$

The same rule applies if sin ax be replaced by cos ax,

i.e., $\frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax$

In general, f(D) will consists of even and odd powers of D. But we can always express f(D) in the form $\phi(D^2) + D\psi(D^2)$

$$\frac{1}{f(\mathrm{D})}\sin ax = = \frac{1}{\phi(D^2) + \mathrm{D}\psi(D^2)}\sin ax$$

$$= \frac{1}{\phi(-a^2) + \mathrm{D}\psi(-a^2)}\sin ax$$

$$= \frac{\phi(-a^2) - \mathrm{D}\psi(-a^2)}{\phi^2(-a^2) - \mathrm{D}^2\psi^2(-a^2)}\sin ax$$

$$= \frac{\phi(-a^2)\sin ax - a\cos ax\psi(-a^2)}{\phi^2(-a^2) + a^2\psi^2(-a^2)}$$

Case 2. Let $\phi(-a^2) = 0. \therefore D^2 + a^2$ is a factor of $\phi(D^2)$.

Hence $\phi(D^2) = (D^2 + a^2)\psi(D^2)$, where $\psi(-a^2) \neq 0$.

$$\therefore \frac{1}{\phi(\mathrm{D}^2)}\sin ax = \frac{1}{(\mathrm{D}^2 + a^2)\psi(\mathrm{D}^2)}\sin ax$$

$$= \frac{1}{\psi(-a^2)}\frac{1}{D^2 + a^2}\sin ax$$

Now, $\frac{1}{\mathrm{D}^2+a^2}\sin ax = \frac{1}{\mathrm{D}^2+a^2}$ Imaginary Part of e^{aix}

as $e^{aix} = \cos ax + i\sin ax$ (by Euler's formula)

$=$ Imaginary Part of $\frac{1}{\mathrm{D}^2+a^2}e^{aix}$

$$= \text{Imaginary Part of } \frac{1}{(\mathrm{D} - ai)(\mathrm{D} + ai)}e^{aix}$$

$$= \text{Imaginary Part of } \frac{1}{(\mathrm{D} - ai)2ai}e^{aix}$$

$= \text{Imaginary Part of } \frac{xe^{aix}}{2ai}$

$$= \text{Imaginary Part of } \frac{-xi}{2a}(\cos ax + i\sin ax)$$
$$= -\frac{x\cos ax}{2a}$$

A similar procedure is adopted in the case when $X = \cos ax$. $\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}$.

Example 1:

Solve $(D^4 + 2D^2n^2 + n^4)y = \cos mx$.

Solution:

To find C.F., solve $(D^4 + 2D^2n^2 + n^4)y = 0$.

The auxiliary equation is $k^4 + 2k^2n^2 + n^4 = 0$.

(Here k is used instead of m as another m occurs in the second member.)

$(k^2 + n^2)^2 = 0$ or $k^2 = -n^2$ twice.
$k = \pm$ in twice.
C.F. $= (A\cos nx + B\sin nx)(C + Dx)$

$\text{P.I.} = \frac{1}{(D^2+n^2)^2}\cos mx = \frac{1}{(-m^2+n^2)^2}\cos mx$

$\therefore y = (A\cos nx + B\sin nx)(C + Dx) + \frac{\cos mx}{(n^2-m^2)^2}$.

Example 2:

Solve $(D^2 - 8D + 9)y = 8\sin 5x$.

Solution:

To find C.F., solve $(D^2 - 8D + 9)y = 0$.

The auxiliary equation is $m^2 - 8m + 9 = 0$.

Solving, $m = 4 \pm \sqrt{7}$.

$\therefore \text{C.F.} = e^{4x}\left(C_1e^{\sqrt{7}x} + C_2e^{-\sqrt{7}x}\right)$.

$$= \frac{1}{D^2 - 8D + 9} 8\sin 5x$$

$$= \frac{8}{-25 - 8D + 9} \sin 5x$$

P.I. $= -\frac{1}{D + 2} \sin 5x = -\frac{D - 2}{D^2 - 4} \sin 5x$

$$= -\frac{5\cos 5x - 2\sin 5x}{-25 - 4}$$

$$= \frac{5\cos 5x - 2\sin 5x}{29}$$

$y =$ C.F. + P.I.

$$y = e^{4x}\left(C_1 e^{\sqrt{7}x} + C_2 e^{-\sqrt{7}x}\right) + = \frac{5\cos 5x - 2\sin 5x}{29}$$

Example 3:

Show that the solution of the differential equation $\frac{d^2y}{dt^2} + 4{:} = \mathrm{A}\sin pt$ which is such that $y = 0$ and $\frac{dy}{dt} = 0$ when $t = 0$ is $y = \frac{\mathrm{A}\left(\sin pt - \frac{1}{2} p \sin 2t\right)}{4 - p^2}$ if $p \neq 2$.

If $p = 2$, show that $y = \frac{\mathrm{A}(\sin 2t - 2t\cos 2t)}{8}$.

To find the C.F., solve $(D^2 + 4)y = 0$ where D stands for $\frac{d}{dt}$ The auxiliary equation is $m^2 + 4 = 0$. Hence $m = \pm 2$. C.F. $= C_1 \cos 2t + C_2 \sin 2t$. (Note that the independent variable is t.)

P.I. $= \frac{1}{D^2 + 4}$ A $\sin pt$

$$= \frac{A}{4 - p^2} \sin pt \text{ if } p^2 \neq 4$$

$\therefore\ y = C_1 \cos 2t + C_2 \sin 2t + \frac{A}{4 - p^2} \sin pt.$

To determine the values of C_1 and C_2, we note that

when $t = 0, y = 0$ and $\frac{dy}{dt} = 0$. $\therefore\ 0 = C_1$.

$$\frac{dy}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t + \frac{Ap}{4 - p^2} \cos pt$$

$$\therefore\ 0 = 2C_2 + \frac{Ap}{4 - p^2} \therefore C_2 = -\frac{Ap}{2(4 - p^2)}$$

Hence $y = \frac{A\left(\sin pt - \frac{1}{2}p\sin 2t\right)}{4-p^2}$.

If $p = 2$, P.I. $= \frac{1}{D^2+4}$ $A\sin 2t$

$=$ Imaginary Part of $\frac{A}{D^2+4}e^{2it}$

$$= \text{Imaginary Part of } \frac{A}{(D+2i)(D-2i)}e^{2it}$$

$$= \text{Imaginary Part of } \frac{A}{4i}te^{2it}$$

$$= \text{Imaginary Part of } -\frac{At}{4}i(\cos 2t + i\sin 2t)$$

$$= -\frac{At\cos 2t}{4}.$$

$$y = C_1\cos 2t + C_2\sin 2t - \frac{At\cos 2t}{4}$$

When $t = 0, y = 0. \therefore 0 = C_1$

$$\frac{dy}{dt} = -2C_1\sin 2t + 2C_2\cos 2t - \frac{A}{4}(\cos 2t - 2t\sin 2t)$$

$$y = \frac{A(\sin 2t - 2t\cos 2t)}{8}$$

Example 4:

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution:

The auxiliary equation is $m^2 - 4m + 3 = 0$.

$\therefore m = 1$ or 3 .

C.F. $= Ae^{x} + Be^{3x}$.

P.I. $= \frac{1}{D^2-4D+3}\sin 3x\cos 2x$

$$= \frac{1}{D^2-4D+3}\frac{\sin 5x + \sin x}{2}$$

$= \frac{1}{-25-4D+3}\frac{\sin 5x}{2} + \frac{1}{-1-4D+3}\frac{\sin x}{2}$ by §4.2. (b)

$$= \frac{2D-11}{-4(4D^2-121)}\sin 5x + \frac{1}{4}\frac{1+2D}{1-4D^2}\sin x$$

$$= \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

$y =$ C.F. + P.I.

$$y = Ae^{x} + Be^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

Example 5:

Solve $(D^2 + 16)y = e^{-3x} + \cos 4x$.

Solution:

The auxiliary equation is $m^2 + 16 = 0$.

$\therefore m = \pm 4i$.

C.F. $= A\cos 4x + B\sin 4x$.

P.I$_1$ corresponding to e^{-3x}

$$= \frac{1}{D^2 + 16} e^{-3x} = \frac{1}{25} e^{-3x}$$

P.I$_2$ corresponding to $\cos 4x = \frac{1}{D^2+16} \cos 4x$

$$= \frac{1}{D^2 + 16} \text{ Real Part of } e^{4ix}$$

$$= \text{ Real Part of } \frac{1}{(D + 4i)(D - 4i)} e^{4ix}$$

$$= \text{ Real Part of } \frac{1}{8i} x e^{4ix}$$

$$= \text{ Real Part of } -\frac{x}{8} i(\cos 4x + i\sin 4x)$$

$$= \frac{x}{8} \sin 4x$$

$$\therefore y = A\cos 4x + B\sin 4x + \frac{1}{25} e^{-3x} + \frac{x}{8} \sin 4x.$$

(c) X is of the form $e^{ax}V$, where Vis any function of x.

$D(e^{ax}\ V) = e^{ax}(D + a)V. \therefore e^{-ax}D(e^{ax}\ V) = (D + a)V$

We may interpret the result thus : the effect of operating on V by $e^{-ax}De^{ax}$ is the same as $D + a$ operating on V .

Symbolically, $(e^{-ax}\mathrm{D}e^{ax})\mathrm{V} = (\mathrm{D}+a)\mathrm{V}$.

Operate on both sides by $e^{-ax}\mathrm{D}e^{ax}$.

$$(e^{-ax}\mathrm{D}e^{ax})(e^{-ax}\mathrm{D}e^{ax}\ \mathrm{V}) = (\mathrm{D}+a)(\mathrm{D}+a)\mathrm{V}$$
$$(e^{-ax}\mathrm{D}^2e^{ax})\mathrm{V} = (\mathrm{D}+a)^2\ \mathrm{V}$$

Again repeat the same operation; we get

$$(e^{-ax}\mathrm{D}^3e^{ax})\mathrm{V} = (\mathrm{D}+a)^3\ \mathrm{V}$$

If the operation be performed n times,

$$(e^{-ax}\mathrm{D}^{\mathrm{n}}e^{ax})\mathrm{V} = (\mathrm{D}+a)^n\ \mathrm{V}$$
$$\therefore\ \mathrm{D}^{\mathrm{n}}(e^{ax}\ \mathrm{V}) = e^{ax}(\mathrm{D}+a)^n\ \mathrm{V}$$

If $f(D)$ is a rational integral function of D,

$$f(\mathrm{D})e^{ax}\ \mathrm{V} = e^{ax}f(\mathrm{D}+a)\mathrm{V}$$

Operate on both sides by $\frac{1}{f(\mathrm{D})}$.

$$e^{ax}\ \mathrm{V} = \frac{1}{f(\mathrm{D})}e^{ax}f(\mathrm{D}+a)\mathrm{V}$$

Denoting $f(\mathrm{D}+a)\mathrm{V}$ by V_1,

$$e^{ax}\frac{1}{f(\mathrm{D}+a)}\mathrm{V}_1 = \frac{1}{f(\mathrm{D})}e^{ax}\ \mathrm{V}_1$$

Hence, reversing, $\frac{1}{f(\mathrm{D})}e^{ax}\cdot\mathrm{X} = e^{ax}\frac{1}{f(\mathrm{D}+a)}\mathrm{X}$.

Example 1:

Solve $(\mathrm{D}^3 - 2\mathrm{D} + 4)y = e^x\cos x$.

Solution:

The auxiliary equation is $m^3 - 2m + 4 = 0$

i.e., $(m+2)(m^2 - 2m + 2) = 0$

Solving, $m = -2$ or $1 \pm i$.

$$\text{C.F.} = Ae^{-2x} + e^x(\text{ B}\cos x + \text{C}\sin x)$$

$$\text{P.I.} = \frac{1}{\text{D}^3 - 2\text{D} + 4} e^x \cos x$$

$$= e^x \frac{1}{(\text{D}+1)^3 - 2(\text{D}+1) + 4} \cos x \text{ by §4.2(c)}$$

$$= e^x \frac{1}{\text{D}^3 + 3\text{D}^2 + \text{D} + 3} \cos x$$

$$= e^x \frac{1}{(\text{D}^2+1)(\text{D}+3)} \cos x$$

$$= e^x \frac{\text{D}-3}{(\text{D}^2-9)(\text{D}^2+1)} \cos x$$

$$= e^x \frac{1}{10(\text{D}^2+1)} (\sin x \div 3\cos x) \text{ by §4.2(b)}$$

$$= \frac{e^x}{10} \{ \text{ Imaginary Part } + 3 \text{ Real Part } \} \text{ of } \frac{1}{\text{D}^2+1} e^{ix}$$

$$= \frac{e^x}{10} \{ \text{ Imaginary Part } + 3 \text{ Real Part } \} \text{ of } \frac{1}{(\text{D}+i)(\text{D}-i)} e^{ix}$$

$$= \frac{e^x}{10} \{ \text{ Imaginary Part } + 3 \text{ Real Part } \} \text{ of } \frac{1}{2i} x e^{ix}$$

$$= \frac{e^x}{10} \{ \text{ Imaginary Part } + 3 \text{ Real Part } \} \text{ of } -\frac{i}{2} (\cos x + i\sin x)$$

$$= \frac{xe^x}{20} (-\cos x + 3\sin x).$$

$y =$ C.F. + P.I.

$$y = Ae^{-2x} + e^x(\text{ B}\cos x + \text{C}\sin x) + \frac{xe^x}{20}(-\cos x + 3\sin x).$$

(d) X is of the form x^m (a power of x), m being a positive integer.

To evaluate $\frac{1}{f(D)} x^m$, raise f(D) to power - 1 and expand in ascending powers of D as far as D^m. These terms in the expansion of $\{f(\text{D})\}^{-1}$ operating on x^m the particular integral required. Examples.

Example 1:

Solve $(\text{D}^3 - \text{D}^2 - \text{D} + 1)y = 1 + x^2$.

Solution:

The auxiliary equation is $m^3 - m^2 - m + 1 = 0$.

Solving, $m = -1$ and $m = 1$ twice.

C.F. $= \mathrm{A}e^{-x} + e^{x}(\mathrm{B} + \mathrm{C}x)$.

$$\begin{aligned}\text{P.I.} &= \frac{1}{(\mathrm{D}-1)^2(\mathrm{D}+1)}(1+x^2)\\ &= (1+\mathrm{D})^{-1}(1-\mathrm{D})^{-2}(1+x^2)\\ &= (1-\mathrm{D}+\mathrm{D}^2)(1+2\mathrm{D}+3\mathrm{D}^2)(1+x^2)\end{aligned}$$

expanding as far as D^2

$$\begin{aligned}&= (1+\mathrm{D}+2\mathrm{D}^2)(1+x^2)\\ &= 5+2x+x^2\\ y &= \text{C.F.} + \text{P.I.}\end{aligned}$$

$$\mathrm{y} = \mathrm{A}e^{-x} + e^{x}(\mathrm{B}+\mathrm{C}x)+(5+2x+x^2)$$

Example 2:

Solve $(\mathrm{D}^4 + \mathrm{D}^3 + \mathrm{D}^2)y = 5x^2 + \cos x$.

The auxiliary equation is $m^4 + m^3 + m^2 = 0$.

Solving, $m = 0$ twice and $m = \frac{-1\pm\sqrt{3}i}{2}$.

$\therefore$ C.F. $= \mathrm{A} + \mathrm{Bx} + \mathrm{e}^{-x/2}\left(\mathrm{C}\cos\frac{\sqrt{3}}{2}x + \mathrm{D}\sin\frac{\sqrt{3}}{2}x\right)$.

P.I. corresponding to $5x^2 = \frac{1}{\mathrm{D}^2(\mathrm{D}^2+\mathrm{D}+1)}5x^2$

$$= \frac{1}{\mathrm{D}^2}(1+\mathrm{D}+\mathrm{D}^2)^{-1}5x^2$$

$$= \frac{1}{D^2}(1-(D+D^2)+(D+D^2)^2-(D+D^2)^3+(D+D^2)^4\}5x^2$$

(It must be noted that we have to expand as far as D^4 in the numerator as D^2 occurs in the denominator.)

Hence the above P.I.

$$= \frac{1}{D^2}(1 - D + D^3 - D^4)5x^2$$

$$= \frac{1}{D^2}5x^2 - \frac{1}{D}5x^2 + (D - D^2)5x^2$$

$= \iint 5x^2(dx)^2 - 5\int x^2dx + (10x - 10)$ as $\frac{1}{D}$

represents an integration

$$= \frac{5x^4}{12} - \frac{5x^3}{3} + 10x - 10$$

P.I. corresponding to $\cos x = \frac{1}{D^2}\frac{1}{(D^2+D+1)}\cos x$

$$= \frac{1}{-1}\frac{1}{(-1 + D + 1)}\cos x$$

$$= -\frac{1}{D}\, cosx = -sinx$$

Hence $y = A + Bx + e^{-x/2}\left(C\cos\frac{\sqrt{3}}{2}x + D\sin\frac{\sqrt{3}}{2}x\right) + \frac{5x^2}{12} - \frac{5x^2}{3} + 10x - 10 - sinx.$

Exercises:

1. Solve $(D^2 - 5D + 6)y = e^{4x}$
2. Solve $(3D^2 + D - 14)y = 13e^{2x}$
3. Solve $(D^2 - 4D - 5)y = e^{3x} + 4\cos 3x$

Unit III

Linear equations of second order: Complete solution in terms of a known integral – Reduction to normal form – Change of independent variable - Applications of first order equations: Flow of water from an orifice – Falling bodies and other rate problems, Free fall under gravity – The Brachistochrone – Fermat and Bernoulli – Simple electric circuits.

Chapter 3: Sections – 3.1 - 3.6

Linear Equations of the Second Order

3.1. Complete solution given a known integral.

If an integral included in the complementary function of the given equation be known, the complete solution can be found in terms of this known integral.

Let $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R}$ .... (1)

be the given equation, where P, Q, R are functions of x.

Let $y = y_1$ be a known integral in the C.F. of (1).

i.e., of $\dfrac{d^2y}{dx^2} + \mathrm{P}\dfrac{dy}{dx} + \mathrm{Q}y = 0$

Putting $y = y_1 v$ in (1), where v is a function of x, we get

$$y_1\frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2\frac{dy_1}{dx} + \mathrm{P}y_1\right) = \mathrm{R} \text{ in virtue of (2).}$$

This is linear in $\frac{dv}{dx}$; hence

$$\frac{dv}{dx} = \frac{c_1}{y_1^2}e^{-\int \mathrm{P}dx} + \frac{e^{-\int \mathrm{P}dx}}{y_1^2}\int \mathrm{R}y_1 e^{\int \mathrm{P}dx}dx$$

Integrating,

$$v = c_2 + c_1\int \frac{e^{-\int \mathrm{P}dx}}{y_1^2}dx + \int\left\{\frac{e^{-\int \mathrm{P}dx}}{v_1^2}\int \mathrm{R}y_1 e^{\int \mathrm{P}dx}dx\right\}dx$$

The solution of (1) is $y = vy_1$, where v has the above value.

It must be noted that this solution includes the given solution and that there are two arbitrary constants.

Note:

Some cases where in simple functions of x, like x and e^x are integrals of the equation

$\mathrm{P}_2\frac{d^2y}{dx^2} + \mathrm{P}_1\frac{dy}{dx} + \mathrm{P}_0 y = 0$ should be noted.

Thus for the above equation

$y = x, y = e^x, y = e^{-x}$, and $y = x^2$

are solution if

$P_1 + P_0 x = 0, P_2 + P_1 + P_0 = 0, P_2 - P_1 + P_0 = 0$ $2P_2 + 2P_1 x + P_0 x^2 = 0$ respectively.

Example 1:

Solve $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$.

Solution:

$$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = e^x$$

As the sum of the coefficients of the first member is zero, e^z is a solution of

$x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0.$

Putting $y = ve^x$, (i) reduces to $x\frac{d^2v}{dx^2} + \frac{dv}{dx} = 1.$

Solving, $\frac{dv}{dx}x = x + c_1$. i.e., $\frac{dv}{dx} = 1 + \frac{c_1}{x}$

Integrating $p = x + c_1 \log x + c_2$.

Hence $y = e^x(x + c_1 \log x + c_2)$.

Example 2:

Solve $x^2\frac{d^2y}{dx^2} - (x^2+2x)\frac{dy}{dx} + (x+2)y = x^3e^x$.

Solution:

$y = x$ is a solution of this equation without the second member, Putting $y = vx$, it reduces to $v_2 - v_1 = e^x$.

Hence $v_1 e^{-x} = x + c_1$ or $v_1 = (x + c_1)e^x$.

Integrating $= (c_1 - 1)e^x + xe^x + c_2$.

$\therefore y = c_2 x + (c_1 - 1)xe^x + x^2 e^x$

3.2. Reduction to the normal form:

Consider $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ………….(1)

Putting $y = y_1 v$, this becomes

$$y_1 \cdot \frac{d^2v}{dx^2} + \frac{dv}{dx}\left(2\frac{dy_1}{dx} + Py_1\right) + v\left(\frac{d^2y_1}{dx^2} + P\frac{dy_1}{dx} + 0y_1\right) = R$$

If y_1 be chosen to satisfy $2\frac{dy_1}{dx} + Py_1 = 0$.

i.e., $y_1 = e^{-\frac{1}{2}\int Pdx}$, then the above equation becomes $\frac{d^2v}{dx^2} + Iv = Pe^{\frac{1}{2}\int Pdx}$

where $\mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dz} - \frac{\mathrm{P}^2}{4}$

(2) is immediately integrable if I be either a constant or $\frac{a}{z^2}$, where a is a constant.

This method is either called reducing (1) to the normal form or removing the first derivative.

Example 1:

Solve $y_2 - 4xy_1 + (4x^2 - 3)y = e^{z^2}$.

Solution:

$y_2 - 4xy_1 + (4x^2 - 3)y = e^{z^2}$.

Here $P = -4x, Q = 4x^2 - 3$ and $P = e^{x^2}$

$y_1 = e^{-\frac{1}{2}\int Pdz} = e^{x^2}$

Putting $y = vy_1, \mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{\mathrm{P}^2}{4} = -1$

and the equation reduces to $\frac{d^2v}{dx^2} - v = 1$.

Hence $v = \mathrm{A}e^x + \mathrm{B}e^{-x} - 1$

$\therefore\ y = e^{x^2}(Ae^x + Be^{-x} - 1)$

Example 2:

$4x^2\frac{d^2y}{dx^2} + 4x^5\frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0.$

Solution:

$4x^2\frac{d^2y}{dx^2} + 4x^5\frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0.$

Divided by x^2

$4\frac{d^2y}{dx^2} + 4x^3\frac{dy}{dx} + \left(x^6 + 6x^2 + \frac{4}{x^2}\right)y = 0.$

Here $P = x^3; Q = \frac{1}{4}\left(x^6 + 6x^2 + \frac{4}{x^2}\right)$

$$y_1 = e^{-\frac{1}{2}\int \mathrm{P}dx} = e^{-\frac{x^4}{8}}$$

$$\mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{\mathrm{P}^2}{4} = \frac{1}{x^2}$$

Hence the equation in v, where $y = vy_1$, is

$$\frac{d^2v}{dx^2} + \frac{1}{x^2}v = 0$$

This is a homogeneous linear equation whose solution is

$\sqrt{x}\ \mathrm{A}\cos\left(\frac{\sqrt{3}}{2}\log x + \mathrm{B}\right)$

$\therefore y = e^{-\frac{x^4}{8}}\sqrt{x}\ \mathrm{A}\cos\left(\frac{\sqrt{3}}{2}\log x + \mathrm{B}\right)$

change of the independent variable

3.3. Change of the independent variable:

Let z be the new independent variable.

Consider $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}y = \mathrm{R}$

$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx};\ \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$ ………….. (1)

The equation (1) transforms into

$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx}\right) + \mathrm{Q}y = \mathrm{R}$ ……….. (2)

We may choose z such that the coefficient of $\frac{dy}{dz}$

i.e., $\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx}$ vanishes. $\therefore\ z = \int\ e^{-\int \mathrm{P}dx}dx$ ………..(3)

Using this value of z, (2) may be come integrable. One particular case where (2) becomes immediately integrable is when $Q = \mu\left(\frac{dz}{dx}\right)^2$, where μ is a constant. Then (2) reduces to

$$\frac{d^2y}{dz^2} + \mu y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

The second case, when (2) becomes integrable, is when

$$Qz^2 = \mu\left(\frac{dz}{dx}\right)^2$$

Then (2) becomes a homogeneous linear equation.

Example 1:

Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + y\cos^2 x = 0.$

Solution:

$\frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + y\cos^2 x = 0.$

Here $P = \tan x$ and $Q = \cos^2 x$.

Choosing z as in $(3), z = \int e^{\int -\tan x dx}dx = \sin x.$

The equation transforms to $\frac{d^2y}{dz^2} + y = 0$ as here

$$Q = \cos^2 x = \left(\frac{dz}{dx}\right)^2$$

$\therefore\ y = \text{A}\cos z + \text{B}\sin z$

$= \text{A}\cos(\sin x) + \text{B}\sin(\sin x)$

Example 2:

Solve: $\frac{d^2y}{dx^2} - \frac{3x+1}{x^2-1}\frac{dy}{dx} + y\left\{\frac{6(x+1)}{(x-1)(3x+5)}\right\}^2 = 0$

Solution:

$$\frac{d^2y}{dx^2} - \frac{3x+1}{x^2-1}\frac{dy}{dx} + y\left\{\frac{6(x+1)}{(x-1)(3x+5)}\right\}^2 = 0$$

Using the notation of (3), if z be the new independent variable, the equation becomes

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left\{\frac{d^2z}{dx^2} + \text{P}\frac{dz}{dx}\right\} + Qy = 0$$

Here $P = -\frac{3x+1}{x^2-1}$ and $Q = \left\{\frac{6(x+1)}{(x-1)(3x-5)}\right\}^2$.

$$\therefore \frac{d^2z}{dx^2} - \frac{dz}{dx}\frac{(3x+1)}{x^2-1} = 0.$$

If $z_1 = \frac{dz}{dx}$, integrating $\log z_1 = \int \frac{3x+1}{x^2-1}dx + C$

$= \int \left(\frac{2}{x-1} + \frac{1}{x+1}\right)dx + C$

$= 2\log(x-1) + \log(x+1) + \log \text{A}.$

$\therefore z_1 = \text{A}(x-1)^2(x+1)$. We can take

$z_1 = x-1)^2(x+1).$

$\therefore\ z = \int\ (x-1)^2(x+1)dx$

$= \frac{(x-1)^3(3x+5)}{12}$ neglecting the constant of integration.

The transformed equation is $\frac{d^2y}{dz^2} \div \frac{Qy}{\left(\frac{dz}{dx}\right)^2} = 0$

i.e., $\frac{d^2y}{dz^2} \div \frac{36y}{(x-1)^6(3x+5)^2} = 0$

i.e., $\frac{d^2y}{dz^2} \div \frac{y}{4z^2} = 0$

$\therefore 4z^2\frac{d^2y}{dz^2} \div y = 0$

Put $u = \log z$ and $\text{D} \equiv \frac{d}{du}$.

$[4D(D-1) \div 1]y = 0$ i.e., $(2D-1)^2 y = 0$

$\therefore y = e^{u/2}(\text{ A} \div \text{B}u)$

$= \sqrt{z}(\text{ A} \div \text{B}\log z)$ where $z = \dfrac{(x-1)^3(2x+5)}{12}$

Applications of First Order Equations:

3.4. Growth, Decay and Chemical Reactions:

When a molecule has a tendency to decompose naturally into smaller molecules at a rate not depending on the presence of other substances, then the number of molecules of this kind decomposing in a unit time will be proportional to the total number present. A chemical reaction of this type is called a first order reaction.

Let, at $t = 0, x_0$ grams of matter be present and decompose in a first order reaction. If x grams. of matter be present at time t, the above principleleads us to the differential equation.

$\frac{dx}{dt} = -kx(k > 0)$ .... (1)

Separating the variables, we get $\frac{dx}{x} = -kdt$.

Integrating, $\log x = -kt + c$.

The initial conditions are $t = 0, x = x_0$.

$\therefore$ $\log x_0 = c$.

Hence $\log x = -kt + \log x_0$

$\Rightarrow \log \dfrac{x}{x_0} = -kt$ or $x = x_0 e^{-kt}$

The positive constant k is called the rate constant for it measures the rate at which decomposition takes place.

An example of the above type of reaction is radioactive decay. It is usual to express the rate of decay in terms of its half-life, i.e., the time required for a given quantity of element to decrease by one-half. If in (2) we put $x = \frac{x_0}{2}$, the half-life, T is given by $\frac{x_0}{2} = x_0 e^{-k\text{T}}$, i.e., $k\text{ T} = \log 2$.

If k or T is known by observation, the other is determined by this equation.

These ideas are fundamental in Geology or Archaeology. Radioactive elements occurring in nature with known half-lives can be used to fix dates for events occurring long ago.

Second-order reaction. Let two chemical substances in solution react together to form a compound. If the reaction is generated by the collision and interaction of the molecules of the substances, the rate of formation of the compound is jointly proportional to the amounts of substances that are untransformed. A chemical reaction of this type is called a second-order reaction and this law of reaction is called the law of mass action.

Let us consider a second-order reaction in which x grams of the compound contain ax grams of the first substance and bx grams of the second, where $a+b=1$. If initially there be a A and b B grams of the first and second substances initially and if $x=0$ when time $t=0$, we shall find x as a function of t.

The law of mass action gives us the equation

$$\frac{dx}{dt}=k(a\,\mathrm{A}-ax)(b\,\mathrm{B}-bx)\ldots(1)$$

$$=kab(\mathrm{A}-x)(\mathrm{B}-x)$$

Separating the variables,

$$\frac{dx}{(\mathrm{A}-x)(\mathrm{B}-x)}=kabdt$$

$$\Rightarrow \frac{1}{\mathrm{A-B}}\left(\frac{1}{\mathrm{B}-x}-\frac{1}{\mathrm{A}-x}\right)dx=kabdt.$$

Integrating, $\frac{1}{\mathrm{A-B}}\log\frac{\mathrm{A}-x}{\mathrm{B}-x}=kabt+\frac{1}{\mathrm{A-B}}\log\mathrm{C}$ (say).

Initial conditions are $\mathrm{t}=0, x=0$.

$$\therefore \mathrm{C}=\frac{\mathrm{A}}{\mathrm{B}}.$$

$$\therefore \frac{\mathrm{A}-x}{\mathrm{B}-x}=\frac{\mathrm{A}}{\mathrm{B}}e^{kabt(\mathrm{A-B})}.$$

Let $\mathrm{B}<\mathrm{A}$; $\frac{\mathrm{A+B}-2x}{\mathrm{A-B}}=\frac{\mathrm{A}e^{kabt(\mathrm{A-B})}+\mathrm{B}}{\mathrm{A}e^{kabt(\mathrm{A-B})}-\mathrm{B}}$

(by component do et dividend)

$$\Rightarrow x=\frac{\mathrm{AB}\left[1-e^{kabt(\mathrm{A-B})}\right]}{\mathrm{A}-\mathrm{B}e^{kabt(\mathrm{A-B})}}$$

If, however $\mathrm{A}=\mathrm{B}$, (1) becomes $\frac{dx}{dt}=kab(\mathrm{A}-x)^2$.

$$\therefore \frac{dx}{(\mathrm{A}-x)^2}=kabdt.$$

Integrating, $\frac{1}{\mathrm{A}-x}=kabt+\mathrm{C}$.

Initially $t=0, x=0; \frac{1}{\mathrm{A}}=\mathrm{C}$.

$$\therefore \frac{1}{\mathrm{A}-x}-\frac{1}{\mathrm{A}}=kabt$$

$$\Rightarrow \frac{x}{(\mathrm{A}-x)}=k\,\mathrm{A}abt.$$

$\therefore x = k\text{Aabt}(\text{A} - x)$

$$x = \frac{k\ \text{A}^2 abt}{1 + k\ \text{A}abt}$$

Example 1:

(i) Suppose that x_0 bacteria are placed in a nutrient solution at time $t = 0$, and that x is the population of the colony at a later time t. If food and living space are unlimited and if, as a consequention at any moment is increasing at a rate proportional to the popultion at the moment, find x as a function of t.

(ii) If, in (i) space is limited and food is supplied at a constant rate, then competition for food and space will act in such a way that ultimately the population will stabilize at a constant level x_1. Assume that under these conditions the population grows at a rate jointly proportional to x and to the difference $x_1 - x$, find x as a function of t.

(i) The differential equation, in the above case, is $\frac{dx}{dt} = kx$.

Separating the variables, $\frac{dx}{x} = kdt$.

Integrating, $\log x = kt + c$.

Initial conditions ; $t = 0, x = x_0$.

$\therefore \log x_0 = c$.

Hence $\log \frac{x}{x_0} = kt$ or $x = x_0 e^{kt}$.

(ii) The differential equation, in this case is $\frac{dx}{dt} = kx(x_1 - x)$.

Separating the variables, $\frac{dx}{x(x_1 - x)} = kdt$

i.e., $\frac{dx}{x_1}\left(\frac{1}{x} + \frac{1}{x_1 - x}\right) = kdt$

Integrating, $\log \frac{x}{x_1 - x} = kx_1 t + \log \mathbf{C}$.

Initially, $t = 0; x = x_0$.

$\therefore \log \frac{x_0}{x_1 - x_0} = \log \text{C}$ whence $\text{C} = \frac{x_0}{x_1 - x_0}$

Hence $\frac{x}{x_1 - x} = \frac{x_0}{x_1 - x_0} e^{x_1 kt}$.

Solving for $x, x = \frac{x_0 x_1}{x_0 + (x_1 - x_0)e^{-kx_1 t}}$.

Example 2:

If, in a culture of yeast, the active ferment doubles itself in three hours, by what ratio will it increase in 15 hours, on the assumption that the quantity increases at a rate proportional to itself?

Solution:

Let the amount of yeast at time t be x. Since the rate of increase varies as itself, we have $\frac{dx}{dt} = kx$, where k is a constant.

i.e., $\frac{dx}{x} = kdt$

$\therefore \log x = kt + \log \mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant.

$\therefore\; x = ce^{kt}$

If the amount of yeast at time $t = 0$ is x_0, then the amount of yeastat $t = 3$ is $2x_0$.

Hence $x_0 = c$ and $2x_0 = ce^{3k} = x_0e^{3k}$.

$\therefore e^{3k} = 2$, i.e., $e^k = 2^{1/3}$

Hence $x = x_0 2^{t/3}$.

The amount of yeast at 15 hours is $x_0 2^5 = 32x_0$.

Hence in 15 hours, it multiplies itself 32 times.

Example 3:

The rate at which one substance combines with another is supposed to be proportional to the amount of the first substance remaining. If there be 15 grams of the first substance when $t = 0$ and 5 grams when $t = 8$ seconds, find how much will be left when $t = 5$ seconds. Also find the value of t when there is one gram left.

Solution:

Let the weight of the first substance be x grams at time t.

We are given $x = 15$ when $t = 0$ and $x = 5$ when $t = 8$.

$\frac{dx}{dt} = -kx$, where $k > 0$

$\therefore \frac{dx}{x} = -kdt$

Integrating, $\log x = -kt + \log C$

i.e., $x = \mathrm{C}e^{-kt}$

When $t=0, x=15 \therefore 15 = Ce^{-k.0}. \therefore C = 15$.

When $t=8, x=5. \therefore 5 = 15e^{-k(8)} \therefore e^{-k} = \left(\frac{1}{3}\right)^{1/8}$

$$\therefore x = 15\left(\frac{1}{3}\right)^{t/8}$$

When $t=5, x = 15\left(\frac{1}{3}\right)^{5/8} = 5(3)^{3/8}$ grams.

When $x=1, t$ is given by $1 = 15\left(\frac{1}{3}\right)^{t/8}$

$$\left(\frac{1}{3}\right)^{t/8} = \frac{1}{15}$$

i.e., $-\frac{t}{8}\log 3 = -\log 15$

$$\therefore\ t = 8\left(\frac{\log 15}{\log 3}\right) \text{ seconds.}$$

Example 4:

A hot body cools in air at a rate proportional to the difference between the temperature of the body and that of the surrounding air. If the air is maintained at $20°$C. and that of the body cools from $100°$C. to $75°$C. in 10 minutes, when will its temperature be $25°$C.? What will be its temperature in half an hour since it started cooling from $100°$C.?

Solution:

Let the temperature of the body at time t minutes be $\theta°$C. Then,

$$\frac{d\theta}{dt} = k(\theta - 20)(k > 0)$$

i.e., $\frac{d\theta}{\theta - 20} = kdt$.

Integrating. $\log(\theta - 20) = kt + \log A$.

i.e., $\theta - 20 = Ae^{kt}, k$ being an arbitrary constant.

When $t = 0, \theta = 100$. When $t = 10, \theta = 75$.

$\therefore 100 - 20 = A = 80$.

$\therefore \theta - 20 = 80e^{kt}$.

$75 - 20 = 80e^{10k}$, i.e., $e^k = \left(\frac{11}{16}\right)^{1/10}$,

$\therefore$ The relation between θ and t becomes

$$\theta - 20 = 80\left(\frac{11}{16}\right)^{t/10}$$

When $\theta = 25{,}5 = 80\left(\frac{11}{16}\right)^{t/10}$

$\therefore \frac{1}{16} = \left(\frac{11}{16}\right)^{t/10}$

$\Rightarrow -\log 16 = \frac{t}{10}(\log 11 - \log 16)$

$\Rightarrow \ t = 10\frac{\log 16}{\log 16 - \log 11}$ minutes.

When $t = 30, \theta - 20 = 80\left(\frac{11}{16}\right)^{30/10}$

$\therefore \theta = 20 + 80\left(\frac{11}{16}\right)^{3}$

$= 46°\text{C}$. (approx.)

Example 5:

A tank contains 1,000 litres of brine in which 400 grams of salt are dissolved. Fresh water runs into the tank at the rate of 8 litres per minute and the mixture (kept uniform by continuous stirring runs out at the same rate). How long will it be before only 200 grams of salt are left in the tank?

Let the amount of salt in the tank at t minutes be x grams

Hence the amount of salt in the tank at time $t + \Delta t$ minutes is $(x + \Delta x)$ grams.

The amount of salt coming into the tank (i.e., input) during Δt time $= 0$ as fresh water only is let in.

Output (i.e., the amount of salt going out of the tank) during Δt time $= \frac{8x}{1000}\Delta t$.

$\therefore \Delta x = -\frac{8x}{1000}\Delta t$, i.e., $\frac{\Delta x}{\Delta t} = -\frac{x}{125}$.

Taking the limit as $\Delta t \to 0$, we have $\frac{dx}{dt} = -\frac{x}{125}$.

Separating the variables, $\frac{dx}{x} = -\frac{dt}{125}$

i.e., $\log x = -\frac{t}{125} + \log C$

or $x = Ce^{-t/125}$

Initial conditions are $t = 0, x = 400. \therefore 400 = C$.

Hence $x = 400e^{-t/125}$

We are required to find t when $x = 200$.

$\therefore \ 200 = 400e^{-t/125}$

$\Rightarrow -\frac{t}{125} = \log\frac{1}{2}$, i.e., $t = 125(\log_e 2)$ minutes.

Example 6:

A tank contains 100 litres of fresh water. 2 litres per minute of brine, run in, each containing 1 gram of salt and the mixture runs out at 1 litre per minute. Find the amount of salt present when the tank contains 150 litres of water.

Solution:

Let the amount of brine the tank at time $t = 100 + t$ litres.

Input in Δt time after time $t = 2\Delta t$.

Output in Δt time after time $t = \frac{x}{100+t}\Delta t$

∴ Accumulation = Input - output

$$\text{i.e., } \Delta x = 2\Delta t - \frac{x}{100+t}\Delta t$$

$$\therefore \frac{\Delta x}{\Delta t} = 2 - \frac{x}{100+t}$$

Proceeding to the limit when $\Delta t \to 0$,

$$\frac{dx}{dt} = 2 - \frac{x}{100+t}.\text{ i.e., } \frac{dx}{dt} + \frac{x}{100+t} = 2$$

This is linear in x.

$\therefore$ I.F. $= e^{\int \frac{dt}{100+t}} = e^{\log(100+t)} = 100 + t.$

$\therefore x(100+t) = \int 2(100+t)dt + C = (100+t)^2 + C.$

Initially $\mathrm{t} = 0, x = 0.$

$\therefore\ 0 = 100^2 + C$ whence $C = -100^2.$

$\therefore x = \frac{(100+t)^2 - 100^2}{100+t} = \frac{t(200+t)}{100+t}.$

When thetank contains 150 litresofbrine, $t = 50$, since the tank contains 100 litres of fresh water at $t = 0$ and the accumulation per minute in the tank is one litre.

Hence when $t = 50, x = \frac{50(200+50)}{100+50} = 83\frac{1}{3}$ grams.

Example 7:

A moth ball whose radius was originally $\frac{1}{4}$ cm. is found to have a radius $\frac{1}{8}$ cm. after 1 month. Assuming that it evaporates at a rate proportional to its surface, find the radius as a function of time. After what time will it disappear altogether?

Solution:

Let $r$cms. be the radius at time t months from start. Then the volume of the ball is Vcu. cm. $= \frac{4}{3}\pi r^3$ and surface area $= 4\pi r^2$ sq.cm.

By the condition of the problem

$$\frac{dV}{dt} = -kdS(k > 0)$$

$$\therefore\ 4\pi r^2 \frac{dr}{dt} = -k4\pi r^2, \text{ i.e., } dr = -kdt.$$

Integrating $= -kt + c$

Initiallyt $= 0, r = \frac{1}{4}$ cm. $\therefore\ c = \frac{1}{4}$ cm. $\therefore\ r = \frac{1}{4} - kt.$

When $t = 1, r = \frac{1}{8}$ cm. $\therefore k = \frac{1}{8}.$

Hence $r = \frac{1}{4} - \frac{1}{8}\ t = \frac{2-t}{8}$ cm.

$r = 0$ when $t = 2$ months.

After one more month, the moth ball disappears completely.

Exercises 1:

1. A certain radioactive salt decomposes at a rate proportional to the amount present at any instant. How much of the salt will be left 300 years hence if 500 mg. that was set aside 50 years ago has been reduced to 450 mg?
2. If the number of bacteria in a quart of milk doubles in four hours, in how much time will the number be multiplied by 4 ? bacteria present.
3. The amount x of a substance present in a certain chemical reaction after time t is given by $\frac{dx}{dt} = k(a - x)(b - x)$, where $k, a, b,$ are constants and x is zero at $t = 0$. If $x = 3$ when $t = 10$ minutes, find the value of x after 20 minutes when $a = 6$ and $b = 9$. What does this solution reduce to when $a = b$?
4. The rate at which two chemical substances are combining is proportional to the amount of the first substance remaining unchanged. If initially there are 20 grams of this substance and two hours later there are only 10 grams, find how much of the substance will be left at the end of four hours?
5. Assuming that a hot body cools in air at a rate proportional to the difference between the temperature of the body and that of the surrounding air, find the temperature of the body after 30 minutes, if its initial temperature be $100°$C. and its temperature after 10 minutes was $75°$C., the temperature of the air remaining steady at $20°$C.

3.5. Flow of water from an orifice:

A vertical tank of uniform cross-section A is filled with water to an initial height h_0. Water flows out through a hole of area a. Find the height of the water h in the tank as a function of the time t.

The volume of water flowing out in time dt is a vdt, where v is the velocity of water at the orifice at time t. The loss of the height in the tank is dh; hence the loss of the volume is Adh.

$$\therefore avdt = -\mathrm{A}dh$$

$$\text{i.e., } \mathrm{A}\frac{dh}{dt} = -av.$$

But the velocity is related to the height by Torricelli's law,

i.e., $v = c\sqrt{2gh}$

The empirical constant c would be unity if there were no obstruction and no " vena contracta" near the orifice; for ordinary holes with sharp edges it is 0.6 .

$$\therefore \mathrm{A}\frac{dh}{dt} = -0.6a\sqrt{2gh}.$$

Examples.

Find the time required to empty a cylindrical tank 1 metre in diameter and 4 metres long through a hole 5 cm. diameter if the tank is initially full and its axis is (a) vertical, and (b)horizontal.

(a) The axis of the tank is vertical.

$$\therefore \mathrm{A}\frac{dh}{dt} = -0.6a\sqrt{2gh}.$$

Here $A = \pi \cdot \frac{1}{4}$ sq. metres

$$a = \pi \cdot \left(\frac{5}{2}\right)^2 = \frac{25}{4}\pi \text{sq. cm.}$$

$$\therefore \frac{\pi}{4}\frac{dh}{dt} \cdot 100^2 = -0.6\frac{25}{4}\pi\sqrt{2(981h)}$$

$$\Rightarrow \frac{dh}{dt} = -14.8\sqrt{h} \text{ (approx.)}$$

$$\therefore \frac{dh}{\sqrt{h}} = -14.8dt$$

Integrating $2\sqrt{h} = -14.8t + C$.

When $t = 0, h = 400$ cm. $\therefore 40 = \mathrm{C}$.

$\therefore 2\sqrt{h} = 40 - 14.8$, i.e., $\sqrt{h} = 20 - 7.4t$.

If t_1 be the time required for emptying the tank, $h = 0, t = t_1$.

Hence $t_1 = \frac{20}{7.4} = 2.7$ seconds (approx.)

(b) The axis of the tank is horizontal.

Let the height of the water level at time t seconds be h metres.

Then A is rectangle with

length 4 metres and breadth $2\sqrt{h-h^2}$ metres.

$\therefore 8\sqrt{h-h^2}\frac{dh}{dt}\cdot 100^2 = -0.6\pi\frac{25}{4}10\sqrt{2(981h)}$.

$$\sqrt{1-h}\frac{dh}{dt} = -\frac{1.5}{32}\sqrt{2(981)}\pi$$

Integrating $\frac{2}{3}(1-h)^{3/2} = \frac{1.5}{32}\sqrt{2(981)}\pi t + C$.

When $t = 0, h = 1$ metre. $\therefore$ C $= 0$.

Hence $(1-h)^{3/2} = \frac{13.5}{64}\sqrt{218}\pi t$.

When $h = 0, t = \frac{64}{13.5\sqrt{218}\pi}$ seconds $= 1.25$ seconds (approx.)

Exercises 2:

1. A cylindrical tank of radius 10 metres and height 10 metres with its axis vertical is full of water but has a leak at the bottom. Assuming that water leaks at a rate proportional to the depth of the water in the tank and that 10 per cent escapes during the first hour, find a formula for the volume of water left in the tank after t hours.
2. A hemispherical bowl of radius R is initially full of water and a small circular hole of radius r is punched in the bottom at time $t = 0$. How long will the bnwl take to empty itself? (Assume Torricelli's law.)
3. Find the time required for a square tank of side 5 metres and depth 10 metres to empty through a circular hole of diameter 5 cm . at the bottom.
4. Into a tank 4 ft . deep with base 10 ft . square water flows at the rate of 24 cu . ft. per minute. Find the time required to fill the tank if at the same time the water leaks out through a circular hole of diameter 2 inches at the bottom.
5. Two open tanks with identical small holes in the bottom drain in the same time. One is a cylinder with vertical axis and the other is a cone with vertex down. If they have equal bases and the height of the cylinder is h, what is the height of the cone?

3.6. Falling bodies and other rate problems:

Free fall under gravity. If y be the distance through which a body falls freely in time t, its equation of motion is $\frac{d^2y}{dt^2} = g$.

On integration, we get the velocity $v = \frac{dy}{dt} = gt + c$.

Suppose, initially, at $t = 0, v = v_0$.

Then $v_0 = c$.

$$\therefore\ v = \frac{dy}{dt} = gt + v_0$$

Integrating $y = \frac{gt^2}{2} + v_0 t + c$.

If, at $t = 0, y = y_0$, we have $y_0 = c$.

$$\because\ y = \frac{gt^2}{2} + v_0 t + y_0$$

If the body falls from rest starting from $y = 0$, then

$v_0 = y_0 = 0$ and hence $v = gt$ and $y = \dfrac{gt^2}{2}$

Eliminating t, $v = \sqrt{2gy}$.

This is a direct deduction from the Principle of conservation of energy, viz., Kinetic energy + Potential energy = Constant.

§ 3.2. Retarded fall. Assuming that air exerts a resistance proportional to the velocity, the differential equation of motion is

$m\frac{d^2y}{dt^2} = mg - k\frac{dy}{dt}$ (m being the mass of the body).

If we put $v = \frac{dy}{dt}, \frac{dv}{dt} = g - \frac{k}{m}v = g - \mathrm{L}v$, where $\mathrm{L} = \frac{k}{m}$.

Separating the variables, $\frac{dv}{g-\mathrm{L}v} = dt$.

Integrating, $-\frac{1}{\mathrm{L}}\log(g - \mathrm{L}v) = t + c, c$ being an arbitrary constant.

If we take the initial conditions $v = 0$ when $t = 0$,

$$-\frac{1}{\mathrm{L}}\log g = c.$$

$$\therefore\ \frac{1}{\mathrm{L}}\log\frac{g}{g - \mathrm{L}v} = t$$

$$\text{or}\ \ v = \frac{g}{\mathrm{L}}(1 - e^{-\mathrm{L}t})$$

As $\mathrm{L} > 0$, when $t \to \infty, v \to \frac{g}{\mathrm{L}}$.

This limiting value of v is called Limiting or Terminal velocity.

From (1), $\frac{dy}{dt} = \frac{g}{L}(1 - e^{-Lt})$.

$$\therefore y = \frac{g}{\mathrm{L}}\left(t + \frac{e^{-\mathrm{L}t}}{\mathrm{L}}\right) + c$$

Taking $y = 0$ when $t = 0, c = -\frac{g}{\mathrm{L}^2}$.

Hence $y = \frac{g}{\mathrm{L}}\left(t + i\frac{e^{-\mathrm{L}t}}{\mathrm{L}} - \frac{1}{\mathrm{L}}\right)$

Example 1:

If the air resistance on a falling body of mass m exerts a retarding force proportional to the square of the velocity, the equation of motion is $\frac{dv}{dt} = g - cv^2$, where $c = \frac{k}{m}$. If $v = 0$ when $t = 0$, find v as a function of t. What is the terminal velocity?

The equation of motion is

$$m\frac{d^2y}{dt^2} = mg - k\left(\frac{dy}{dt}\right)^2$$

Putting $\frac{dy}{dt} = v$ and $c = \frac{k}{m}$, it becomes $\frac{dv}{dt} = g - cv^2$

$$\therefore \frac{dv}{g - cv^2} = dt \text{ or } \frac{dv}{\frac{g}{c} - v^2} = cdt$$

Integrating, $\frac{1}{2}\left(\frac{c}{g}\right)^{1/2} \log\frac{\left(\frac{g}{c}\right)^{1/2}+v}{\left(\frac{g}{c}\right)^{1/2}-v}$

arbitrary constant.

Initially $\mathrm{t} = 0$ and $v = 0.0 = \mathrm{A}$.

$$\therefore \log\frac{\left(\frac{g}{c}\right)^{1/2} + v}{\left(\frac{g}{c}\right)^{1/2} - v} = 2\sqrt{gct}$$

$$\text{or } \frac{\left(\frac{g}{c}\right)^{1/2} + v}{\left(\frac{g}{c}\right)^{1/2} - v} = e^{2\sqrt{gct}}$$

$$\frac{v}{\left(\frac{g}{c}\right)^{1/2}} = \frac{e^{2\sqrt{gct}} - 1}{e^{2\sqrt{gct}} + 1} = \tanh(\sqrt{gct})$$

When $t \to \infty, v \to \left(\frac{g}{c}\right)^{1/2}$,i.e., the terminal velocity. Ex.2. Inside the earth, the force of gravity is proportional to

the distance from the center. If a hole be drilled from pole to

pole and a rock is dropped in the hole pole and a rock is dropped in the hole, with what velocity will it reach

the centre?

At distance x from the centre of the earth, the acceleration is $\frac{d^2x}{dt^2}$ and it varies as x

$\therefore$ The equation of motion is $\frac{d^2x}{dt^2} = -\mu x (\mu > 0)$.

If the velocity be $v = \frac{dx}{dt}$, it becomes $v\frac{dv}{dx} = -\mu x$.

Integrating, $\frac{v^2}{2} = -\frac{\mu x^2}{2} + C$.

At the surface of the earth, $x = \mathrm{R}$ (radius of the earth) and $v = 0$.

$\therefore C = \frac{\mu \mathrm{R}^2}{2}$

Hence $v^2 = \mu(\mathrm{R}^2 - x^2)$

At the centre of the earth, $x = 0, v = \sqrt{\mu}\mathrm{R}$

To determine μ, we note that at the surface of the earth, $x = \mathrm{R}$ and $\frac{d^2x}{dt^2}$ in magnitude is g.

$\therefore \mu\mathrm{R} = g$

Hence, from (1), $v = \mathrm{R}\left(\frac{\mathrm{g}}{\mathrm{R}}\right)^{1/2}$

$= \sqrt{gR}$

Exercises 3:

1. A torpedo is travelling at a speed of 60 km /hour at the moment it runs out of fuel. If the water resists its motion with a force proportional to the speed and if 1 km. of travel reduces its speed to 30 km /hour, how far will it coast?
2. If the retardation caused by the resistance to the motion of a train is $a + bv^2$, where v is the velocity, show that it will come to rest from velocity v in a distance $\frac{1}{2b}\log\left(1 + \frac{bv^2}{a}\right)$.
3. The acceleration of a moving particle being proportional to the cube of the velocity and negative, find the distance passed over in time t, theinitial velocitybeing v_0 and thedistancebeingmeasured from the position of the particle at time $t = 0$.

4. If a projectile fired upward from the surface of the earth is to keep travelling indefinitely in space, show that its initial velocity

UNIT IV:

Partial differential equation: Formation of PDE by Eliminating arbitrary constants and arbitrary functions – Complete integral – Singular integral – General integral – Lagrange's Linear Equations.

Chapter 4: Sections – 4.1 to 4.4

Partial Differential Equations of the First Order:

4.1. Introduction:

We now consider equations in which the number of independent variables is two or more and only one independent variable. We usually denote this by z and the independent variables by x and y if there be two, if there be n independent variables, we shall call them $x_1, x_2, x_3 \ldots \ldots x_n$. The partial derivatives $\frac{\delta z}{\delta x}, \frac{\delta z}{\delta y}$ are denoted by p and q while, in the latter case $\frac{\delta z}{\delta x_1}, \frac{\delta z}{\delta x_2}, \ldots \ldots \frac{\delta z}{\delta x_n}$ are represented by $p_1, p_2, \ldots \ldots p_n$ respectively.

Partial differential equations are those which involve one or more partial derivatives. The order of a partial differential equation is determined by the highest order of the partial derivative occurring in it. We consider only partial differential equations of the first order.

4.2. Classification of Integrals:

Let the partial differential equation be

F(x, y, z, p, q)=0 …………(1)

Let the solution of this be $\phi(x, y, z, a, b) = 0$ ………..(2)

Where a and b are arbitrary constants.

The solution (2) which contains as many arbitrary constants as there are independent variables is called the complete integral of (1).

A particular integral of (1) is that got by giving particular values to a and b in (2)

Singular Integral:

The eliminant of a and b between

$\phi(x, y, z, a, b) = 0$

$\frac{\delta \phi}{\delta a} = 0$ and $\frac{\delta \phi}{\delta b} = 0$

When it exists, is called the singular integral.

Geometrically, this includes the envelope of the two parameter surfaces represented by the complete integral (2) of (1). The two parameters occurring in (2) are a and b.

The locus of all points whose coordinates with the corresponding values of p and q satisfy (1) is the doubly infinite system of surfaces represented by(2).As the envelope of these surfaces touches at each point one member of the system (2) the coordinates of any point of the envelope and the associated p and q satisfy (1) and is thus a solution of (1). This is a singular solution as we cannot deduce this from (2) by giving any values of a and b.

General integral.

In(2), we shall assume an arbitrary relation between a and b of the form $b = f(a)$.

Then (2) becomes $\phi[x, y, z, a, f(a)] = 0$.

Differentiating this partially with respect to a,

$$\frac{\delta\phi}{\delta a} + \frac{\delta\phi}{\delta b} f'(a) = 0$$

The eliminant of a between these two equations is called the general integral of (1).

The above two equations represent a curve, viz., the curve of intersection of two consecutive surfaces of the system $\phi\{x, y, z, a, f(a)\} = 0$ for a particular value of a. The envelope of the family of the surfaces touches them along this curve, which is called the characteristic of the envelope. Thus the general integral represents the envelope of a family of surfaces, considered as composed of its characteristics.

Note.

When the singular integral is formed, it is necessary to verify whether the eliminant of a and b between

$\phi = 0, \frac{\delta\phi}{\delta a} = 0$ and $\frac{\delta\phi}{\delta b} = 0$ satisfies (1).

As in (2), this eliminant may include extraneous loci such as locus of conical points and double lines which are not solutions in general of (1)

4.3. Derivation of partial differential equations:

Partial differential equations can be derived either by the elimination of (1) arbitrary constants from a relation between x, y, z or (ii) of arbitrary functions of these variables.

1. By elimination of constants:

Let $\phi(x, y, z, a, b) = 0$ ………..(1)

be a relation between x, y, z involving two arbitrary constants and b.

Differentiating (1) with respect to x and y partially, we get

$\frac{\delta\phi}{\delta x} + \frac{\delta\phi}{\delta z} p = 0$ ………… (2)

$\frac{\delta\phi}{\delta y}+\frac{\delta\phi}{\delta z}q=0$ ………… (3)

Eliminating a and b, we have a partial differential equation of the first order of the form $F(x,y,z,p,q)=0$.

Here the number of constants to be eliminated is equal to the number of independent variables and an equation of the first order results. If the number of constants to be eliminated is greater than the number of independent variables, equations of the second and higher derivatives are deduced.

Example 1:

Eliminate a and b from $z=(x+a)(y+b)$

Solution:

$z=(x+a)(y+b)$ …….. (1)

Equation (1) , Differentiating with respect to x and y partially,

$p=y+b$ and $q=x+a$

Eliminating a and $b, z=pq$.

Exercises 1:

1.Eliminate a and b from $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1$.

2. Eliminate a and b from $z=ax+by+a$.

3. Eliminate h and k from the relation$(x-h)^2+(y-k)^2+z^2=r^2$

4.Eliminate a and b from

(i) $2z=(ax+y)^2+b$

(ii) $ax^2+by^2+z^2=1$.

2. By elimination of an arbitrary function.

Let u and v be any two functions of x,y,z and be connected by an arbitrary relation

$\phi(u,v)=0$ ….……(1)

By eliminating ϕ, we shall form a partial differential equation and show that this is linear, i.e., of the first degree in p and q.

Differentiating(1) partially with respect to x and y,

$$\frac{\delta\phi}{\delta u}\left(\frac{\delta u}{\delta x}+\frac{\delta u}{\delta z}p\right)+\frac{\delta\phi}{\delta v}\left(\frac{\delta v}{\delta x}+\frac{\delta v}{\delta z}p\right)=0$$
$$\frac{\delta\phi}{\delta u}\left(\frac{\delta u}{\delta y}+\frac{\delta u}{\delta z}q\right)+\frac{\delta\phi}{\delta v}\left(\frac{\delta v}{\delta y}+\frac{\delta v}{\delta z}q\right)=0.$$

Eliminating $\frac{\delta\phi}{\delta u}$ and $\frac{\delta\phi}{\delta v}$, we have

$(u_x + u_z p)(v_y + v_z q) = (u_y + u_z q)(v_x + v_z p)$

where $u_x = \frac{\delta u}{\delta x}, u_y = \frac{\delta u}{\delta y}$, etc.

This equation can be put in the form $\mathrm{P}p + \mathrm{Q}q = \mathrm{R}$ where

$\mathrm{P} = u_y v_z - u_z v_y, \mathrm{Q} = u_z v_x - u_x v_z$ and $\mathrm{R} = u_x v_y - u_y v_x$ Examples.

Example 1:

Eliminate the arbitrary function from $z = f(x^2 + y^2)$

Solution:

$z = f(x^2 + y^2)$ …………. (1)

Differentiating(1) partially with respect to x and y,

$p = f'(x^2 + y^2)2x$ and $q = f'(x^2 + y^2)2y$

Eliminating $f'(x^2 + y^2)$ between the latter two equations

$py = qx.$

Example 2:

Eliminate the arbitrary function from $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$

Solution:

$f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$ ……….. (1)

Solving, $x^2 + y^2 + z^2 = \mathrm{F}(z^2 - 2xy)$.

Differentiating(1) partially with respect to x and y,

$2x + 2zp = \mathrm{F}'(z^2 - 2xy)(2zp - 2y)$ ……….. (2)

$2y + 2zq = \mathrm{F}'(z^2 - 2xy)(2zq - 2x)$ ………… (3)

Dividing (2) by (3) to eliminate F',

$\frac{x + zp}{y + zq} = \frac{zp - y}{zq - x}$ or $z(p - q) = y - x$

Example 3:

Eliminate f and ϕ from the relation $z = f(x + ay) + \phi(x - ay)$

Solution:

$z = f(x + ay) + \phi(x - ay)$ ……….. (1)

Differentiating (1) partially with respect to x and y,

$p = f'(x + ay) + \phi'(x - ay)$ … … … (2)

$q = af'(x + ay) - a\phi'(x - ay)$ … … … …. (3)

Differentiating (2) and (3) again, with respect to x and y respectively,

$$\frac{\delta p}{\delta x} = \frac{\delta^2 z}{\delta x^2} = f''(x+ay) + \phi''(x-ay)$$

$$\text{and } \frac{\delta q}{\delta y} = \frac{\delta^2 z}{\delta y^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$$

$$= a^2 \frac{\delta^2 z}{\delta x^2}.$$

Hence the resulting equation is $r - a^2 t = 0$, where

$$r = \frac{\delta^2 z}{\delta x^2} \text{ and } t = \frac{\delta^2 z}{\delta y^2}$$

Exercises 1:

Eliminate the arbitrary functions from

1. $z = e^y f(x+y)$.
2. $z = (x+y)f(x^2 - y^2)$
3. $ax + by + cz = f(x^2 + y^2 + z^2)$.
4. $z = f\left(\frac{xy}{z}\right)$.
5. $f(x^2 + y^2, z - xy) = 0$
6. $f(x+y+z) = xyz$.
7. $z = f(x+y)\phi(x-y)$.

10. $z = f(y+ax) + x\phi(y+ax)$.

4.4. Lagrange's method of solving the linear equation:

Consider the equations $u = a$ and $v = b$, where a and b are arbitrary constants. By eliminating a and b, we form the differential equations corresponding to them.

$$\frac{\delta u}{\delta x}dx + \frac{\delta u}{\delta y}dy + \frac{\delta u}{\delta z}dz = 0,$$

$$\text{and } \frac{\delta v}{\delta x}dx + \frac{\delta v}{\delta y}dy + \frac{\delta v}{\delta z}dz = 0.$$

$$\therefore \frac{dx}{u_y v_z - u_z v_y} = \frac{dy}{u_z v_x - u_x v_z} = \frac{dz}{u_x v_y - u_y v_x}$$

i.e., $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

We have seen that the elimination of the arbitrary function ϕ from $\phi(u, v) = 0$ leads to the linear partial differential equation $Pp + Qq = R$

Thus Lagrange's method of solving the linear equation $Pp + Qq = R$ is as follows:-

Write down the subsidiary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Let the twoindependentintegrals of theseordinarydifferential equations be $u = a$ and $v = b$. Then the solution of the given equation is $\phi(u, v) = 0$, where ϕ is an arbitray function.

$\phi(u, v) = 0$ is called the General Integral of Lagrange's linear equation.

Corollary 1:

This method can be extended to the case of the linear equation of n independent variables.

Consider the equation

$P_1 p_1 + P_2 p_2 + \cdots P_n p_n = R$

From the subsidiary equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

Let n independent integrals of these be

$u_1 = a_1, u_2 = a_2, \dots u_n = a_n$

Then $\phi(u_1, u_2, \dots u_n) = 0$ is a solution of the given equation, where ϕ denotes an arbitrary function.

Note:

The above relation $\phi(u, v) = 0$ or $\phi(u_1 \dots u_n) = 0$ contains all the integrals of the equation which are not of the type called singular.

Corollary 2:

When either $u = a$ or $v = b$ involves z, it is an integral ofthedifferential equation.

$\phi(u, v) = 0$ can be written as $u = f(v)$, where f is arbitrary. We can take $f(v) = av^\circ$, where a is an arbitrary constant; thus the solution reduces to $u = a$.

Example 1:

Solve $(y + z)p + (z + x)q = x + y$.

Solution:

$(y + z)p + (z + x)q = x + y$

The subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{d(\sum x)}{2\sum x}$$

They are also equivalent to

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dz - dx}{x - z} = \frac{\sum dx}{2\sum x}$$

Taking the first two ratios and integrating, $\frac{x-y}{y-z} = a$.

Taking the first and last ratios and integrating,

$$(x-y)^2 \sum x = b$$

Hence the solution required is

$$\phi\left[\frac{x-y}{y-z}, (x-y)^2 \Sigma x\right] = 0 \text{ where } \phi \text{ is arbitrary}$$

Example 2:

Solve $px(y^2+z) - qy(x^2+z) = z(x^2-y^2)$.

Find the surface that contains the straight line $x + y = 0, z = 1$.

Solution:

$$px(y^2+z) - qy(x^2+z) = z(x^2-y^2)$$

The subsidiary equations are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$
$$= \frac{xdx + ydy}{z(x^2-y^2)}$$

Hence taking the last two ratios and integrating, $x^2 + y^2 = 2z + a$

The subsidiary equations can also be written as

$$\frac{\frac{dx}{x}}{y^2+z} = \frac{\frac{dy}{y}}{-(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$
$$= \frac{\frac{dx}{x} + \frac{dy}{y}}{y^2-x^2}$$

Taking the last two ratios, $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$.

Integrating, $\log xyz = \log b$. $\therefore xyz = b$

The solution is $\phi(x^2 + y^2 - 2z, xyz) = 0$.

$$x^2 + y^2 - 2z = f(xyz)$$
$$(x+y)^2 - 2xy - 2z = f(xyz)$$

$x + y = 0, z = 1$ lies on this if $-2(xy+1) = f(xy)$

$\therefore f(xyz) = -2(xyz+1)$

$\therefore$ The desired integral surface is

$$x^2 + y^2 - 2z = -2(xyz+1)$$

Example 3:

Determine the surface which satisfies the differential equation $(x^2 - a^2)p + (xy - \alpha z\tan\alpha)q = xz - ay\cot\alpha$

and passes through the curve $x^2 + y^2 = a^2, z = 0$.

The subsidiary equations are

$$\frac{dx}{x^2 - a^2} = \frac{dy}{xy - az\tan\alpha} = \frac{dz}{xz - ay\cot\alpha}$$

$$= \frac{zdy - ydz}{a(-z^2\tan\alpha + y^2\cot\alpha)}$$

$$= \frac{d\left(\frac{y}{z}\right)}{a\cot\alpha\left(-\tan^2\alpha + \frac{y^2}{z^2}\right)}$$

Taking the first and last ratios and integrating,

$$\log\frac{x-a}{x+a} = \log\frac{\frac{y}{\frac{y}{z}} - \tan\alpha}{\frac{y}{z} + \tan\alpha} + \log A$$

$$\text{or } \frac{x-a}{x+a}\cdot\frac{y + z\tan\alpha}{y - z\tan\alpha} = A$$

where A is an arbitrary constant

The subsidiary equations can again be written as

$$\frac{dx}{x^2 - a^2} = \frac{dy}{xy - az\tan\alpha} = \frac{dz}{xz - ay\cot\alpha}$$

$$= \frac{ydy\cot\alpha - zdz\tan\alpha}{x(y^2\cot\alpha - z^2\tan\alpha)}$$

Taking the first and the last ratios and integrating,

$$\log(x^2 - a^2) = \log(y^2\cot\alpha - z^2\tan\alpha) + \log B$$

or $\frac{x^2-a^2}{y^2\cot\alpha - z^2\tan\alpha} = B$ …………… (2)

where B is an arbitrary constant.

Hence the solution of the given partial differential equation is

$$\frac{x^2 - a^2}{y^2\cot\alpha - z^2\tan\alpha} = f\left(\frac{x-a}{x+a}\cdot\frac{y + z\tan\alpha}{y - z\tan\alpha}\right)$$

where f isarbitrary.

If this surface is to pass through the circle

$x^2 + y^2 = a^2; z = 0$

then $\dfrac{-y^2}{y^2 \cot\alpha} = -\tan\alpha = f\left(\dfrac{x-a}{x+u}\right)$

i.e., $f =$ constant.

Hence the required solution is

$x^2 - a^2 = -(y^2\cot\alpha - z^2\tan\alpha)\tan\alpha$

i.e., $x^2 + y^2 - a^2 = z^2\tan^2\alpha$

Example 4:

Find the integral surface of $x^2p + y^2q + z^2 = 0$ whinet passes through the hyperbola $xy = x + y; z = 1$.

Solution:

$x^2p + y^2q + z^2 = 0$

The subsidiary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Integrating, $\frac{1}{x} + \frac{1}{z} = a$ and $\frac{1}{y} + \frac{1}{z} = b$.

$\therefore$ The solution is $\frac{1}{z} + \frac{1}{x} = f\left(\frac{1}{y} + \frac{1}{z}\right)$, where f is arbituary

If this surface is to pass through the hyperbola $xy = x + x\ z = 1$, we must have

$$1 + \frac{1}{x} = f\left(\frac{1}{y} + 1\right)$$

From $xy = x + y$, we get $1 = \frac{1}{x} + \frac{1}{y}$.

$$\therefore 1 + \frac{1}{x} = 2 - \frac{1}{y} = 3 - \left(1 + \frac{1}{y}\right)$$

$$\therefore f\left(\frac{1}{y} + 1\right) = 3 - \left(1 + \frac{1}{y}\right)$$

Hence the required surface is

$$\frac{1}{z} + \frac{1}{x} = 3 - \left(\frac{1}{y} + \frac{1}{z}\right)$$

UNIT V:

Special methods – Standard forms.

Chapter 5: Sections: 5.1 - 5.5

5. Special methods and Standard forms:

5.1. Standard I:

Equations, in which the variables do not occur explicitly, can be written in the form

$F(p, q) = 0$.

A solution of this is $z = ax + by + c$, where a and b are con. nected by $F(a, b) = 0$. Solving this for $b, b = f(a)$.

Hence the complete integral is

$z = ax + yf(a) + c$

The singular integral is obtained by eliminating a and c between

$$z = ax + yf(a) + c$$
$$0 = x + yf'(a)$$
$$0 = 1.$$

The last equation is absurd and shows that there is not singular integral in the case.

To obtain the general integral, we assume an arbitrary relation $c = \phi(a)$. Then

$z = ax + yf(a) + \phi(a)$.

Differentiating partially with respect to a,

$0 = x + yf'(a) + \phi'(a)$.

The eliminant of a between these equations is the general integral.

Note:

The singular and general integrals must be indicated in every equation be sides the complete integral. Then only the equation besides is said to be the completely solved.

Example 1:

$p^2 + q^2 = npq$.

Solution:

Let the solution be $z = ax + by + c$, where

$a^2 + b^2 = nab$.

Solving, $b = \frac{a\left(n \pm \sqrt{n^2-4}\right)}{2}$.

∴ The complete integral is $z = ax + \frac{ya}{2}\left(n \pm \sqrt{n^2 - 4}\right) + c$.

Differentiating partially with respect to c, we find that there is nosingular integral, as we get an absurd result.

To find the general integral, put $c = f(a)$.

$$z = ax + \frac{ay}{2}\left(n + \sqrt{n^2 - 4}\right) + f(a)$$

Differentiating partially with respect to a,

$$0 = x + \frac{y}{2}\left(n + \sqrt{n^2 - 4}\right) + f'(a).$$

The eliminant of a between the equations is the general integral.

Example 2:

Prove that the characteristics of $q = 3p^2$ that pass through the point $(-1,0,0)$ generate the cone $(x + 1)^2 + 12yz = 0$.

Solution:

Let the complete integral of $q = 3p^2$ be $z = ax + by + c$ where $b = 3a^2$. Hence it is $z = ax + 3a^2y + c$.

The general integral is the locus of all characteristics.

To find the general integral, let us put $c = f(a)$.

Then $z = ax + 3a^2y + f(a)$.

If this passes through $(-1,0,0), f(a) = a$.

The complete integral is $z = a(x + 1) + 3a^2y$.

Differentiating partially with respect to a, $0 = x + 1 + 6ay$.

Eliminating a, the locus of the characteristics is the cone

$(x + 1)^2 + 12yz = 0$

Exercise 1:

Solve the following equations:

1. $pq = k$.
2. $3p^2 - 2q^2 = 4pq$
3. $p^2 + q^2 = 4$.
4. $pq + p + q = 0$.
5. $q^2 - 3q + p = 2$.
6. Obtain the complete integral of $p^2 + q^2 = c^2$ in the form $z = cx\cos\alpha + cy\sin\alpha + b$ and show that $z^2 = c^2(x^2 + y^2)$ is a particular case of ageneralintegral.

7. $\sqrt{p}+\sqrt{q}=1.$

5.2. Standard II:

Only one of the variables x, y, z explicitly. Such equations can be written in one of the forms

$\mathrm{F}(x,p,q)=0; \mathrm{F}(y,p,q)=0; \mathrm{F}(z,p,q)=0$

(i) Let us consider the form $\mathrm{F}(x,p,q)=0$.

Since z is a function of x and y

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$
$$=pdx+qdy$$

Let us assume that $q=a$.

The equation becomes $\mathrm{F}(x,p,a)=0$.

Solving this for p, we get $\mathrm{p}=\phi(x,a)$.

$\therefore dz=\phi(x,a)dx+ady$.

$$\therefore z=\int \phi(x,a)dx+ay+b.$$

This contains two arbitrary constants a and b and hence it is a completeintegral.

(ii) Let us consider the form $\mathrm{F}(y,p,q)=0$.

Let us assume that $p=a$.

$\therefore \mathrm{F}(y,a,q)=0.$

$\therefore q=\phi(y,a).$

Hence $dz=adx+\phi(y,a)dy$.

$\therefore z=ax+\int\phi(y,a)dy+b$ which is a complete integral.

(iii) Let us consider the equation $\mathrm{F}(z,p,q)=0$.

Let us assume that $q=ap$.

Then this equation becomes $\mathrm{F}(z,p,ap)=0$

i.e., $p=\phi(z,a)$

Hence $dz=\phi(z,a)dx+a\phi(z,a)dy$

i.e., $\frac{dz}{\phi(z,a)}=dx+ady$

i.e., $\int\frac{dz}{\phi(z,a)}=x+ay+b$ which is a complete integral.

Example 1:

Solve (i) $q=xp+p^2$

(ii) $p=y^2q^2$

(iii) $p(1+q^2)=q(z-1)$.

Solution:

(i) $q = xp + p^2$

Let $q = a$. Then $a = xp + p^2$ i.e., $p^2 + xp - a = 0$

$$\therefore p = \frac{-x \pm \sqrt{x^2+4a}}{2}$$

Hence $dz = \frac{-x\pm\sqrt{x^2+4a}}{2}dx + ady$

$$\therefore\ z = \int \frac{-x \pm \sqrt{x^2+4a}}{2}dx + ay + b$$

$$= -\frac{x^2}{4} \pm \left\{\frac{x}{4}\sqrt{4a+x^2} + a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right)\right\} + ay + b$$

(ii) $p = y^2q^2$.

Let $p = a^2$. Then $q = \pm\frac{a}{y}$.

Hence $dz = a^2dx \pm \frac{a}{y}dy$.

$\therefore\ z = a^2x \pm a\log y + b$.

(iii) $p(1+q^2) = q(z-1)$.

Let $q = ap$. Then $p(1+a^2p^2) = ap(z-1)$.

$\therefore\ 1 + a^2p^2 = a(z-1)$

$$\text{i.e., } p = \pm\frac{\sqrt{az-a-1}}{a}$$

Hence $dz = \pm\frac{\sqrt{az-a-1}}{a}dx \pm a\frac{\sqrt{az-a-1}}{a}dy$

$$\text{i.e., } \pm\frac{adz}{\sqrt{(az-a-1)}} = dx + ady$$

$$\text{i.e., } \pm\int\frac{adz}{\sqrt{(az-a-1)}} = x + ay + b$$

$$\text{i.e., } \pm 2\sqrt{az-a-1} = x + 2y + b$$

Exercises 2:

Solve:

1. $z(p^2+q^2+1) = a^2$.
2. $p^2 = z^2(1-pq)$
3. $p(1+q) = qz$. (B.Sc.1994)
4. $p(1+q^2) = q(z-a)$.

5. $pq = z$.
6. $p = 2qx$.

5.3. Standard III:

Equations of the form $f_1(x,p) = f_2(y,q)$. In this form the equation is of the first order and the variables are separable. In this equation z does not appear. We shall assume a tentative solution that each of these quantities is equal to a.

Solving $f_1(x,p) = a, p = \phi_1(a,x)$

Solving $f_2(y,q) = a, q = \phi_2(a,y)$.

Hence $dz = \phi_1(a,x)dx + \phi_2(a,y)dy$.

$$\therefore\ z = \int \phi_1(a,x)dx + \int \phi_2(a,y)dy + b$$

which is a complete integral.

Example 1:

Solve the equation $p + q = x + y$.

We can write the equation in the form $p - x = y - q$

Let $p - x = a$. Then $y - q = a$.

Hence $p = x + a, q = y - a$.

$$\therefore dz = (x+a)dx + (y-a)dy$$

$$\therefore z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b$$

There is no singular integral and the general integral is found as usual.

Example 2:

$q(p - \sin x) = \cos y$

Let $p - \sin x = \frac{\cos y}{q} = a$.

$\therefore p = a + \sin x, q = \frac{\cos y}{a}$.

$\therefore dz = (a + \sin x)dx + \frac{\cos y}{a}dy$.

$$\therefore\ z = ax - \cos x + \frac{\sin y}{a} + b$$

Obviously there is no singular integral as partial differentiation with respect to b leads to an absurd result.

To get the general integral, assume $b = f(a)$ where f is arbitrary.

Then $z = -\cos x + ax + \frac{\sin y}{a} + f(a)$

Differentiating with respect to a partially $0 = x - \frac{\sin y}{a^2} + f'(a)$

The eliminant of a between (i) and (ii) represents the general integral.

Exercises 3:

Solve

1. $p^2 + q^2 = x - y.$
2. $y^2(p^2 - 1) = x^2q^2$
3. $\sqrt{p} + \sqrt{q} = \sqrt{x}$
4. $q = xp + p^2.$
5. $pq = xy$
6. $\sqrt{p} + \sqrt{q} = 2x$
7. $p^2 - y^3q = x^2 - y^2$

5.4. Standard IV:

Clairant's form.

This is of the form $z = px + qy + f(p, q)$

The solution of the equation is $z = ax + by + f(a, b)$ equation and $q = b$ can easily be verified to satisfy the given equation

Example 1:

Solve $z = px + qy + \sqrt{1 + p^2 + q^2}$

The complete integral is obviously

$z = ax + by + \sqrt{1 + a^2 + b^2}$

To find the singular integral, differentiating partially with respect to a and b

$$x + \frac{a}{\sqrt{(1 + a^2 + b^2)}} = 0$$

and $y + \frac{b}{\sqrt{(1 + a^2 + b^2)}} = 0$

Eliminating a and b the singular integral is

$x^2 + y^2 + z^2 = 1$

To find the general integral, assume $b = f(a)$ where f is arbitrary.

Differentiate partially with respect to a and eliminate a between the two equations.

Exercises 4:

Solve

1. $z = px + qy + pq$.
2. $z = px + qy + 2\sqrt{pq}$.
3. $z = px + qy + \frac{p}{q} - p$ and clarify the following integrals of this equation

   $z = 2x + 4y, yz = 1 - x$ and $x^2 + 4yz = 0$.
4. $(1 - x)p + (2 - y)q = 3 - z$.
5. $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$.

5.5. Equations reducible to the standard forms:

Many non-linear partial differential equations of the first order do not fall to any of the above four standard forms. Sometimes however, it is possible to make a change of variable which will reduce a given equation to one of the above four forms.

(i) If ($x^m p$) and $(y^n q)$ occur in the partial differential equation as in $\mathrm{F}(x^m p, y^n q) = 0$ or in $\mathrm{F}(z, x^m p, y^n q) = 0$.

(a) Put $x^{1-m} = \mathrm{X}$ and $y^{1-n} = \mathrm{Y}$ if $m \neq 1, n \neq 1$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \mathrm{X}}\frac{\partial \mathrm{X}}{\partial x} = \frac{\partial z}{\partial \mathrm{X}}(1 - m)x^{-m}$$

$\therefore\ x^m p = (1 - m)\frac{\partial z}{\partial \mathrm{X}} = (1 - m)\mathrm{P}$ where $\mathrm{P} = \frac{\partial z}{\partial \mathrm{X}}$.

Similarly $y^n q = (1 - n)Q$, where $\frac{\partial z}{\partial \mathrm{Y}} = \mathrm{Q}$.

Then the equation reduces to $\mathrm{F}(\mathrm{P}, \mathrm{Q}) = 0$ or to $\mathrm{F}(z, \mathrm{P}, \mathrm{Q}) = 0$.

(b) If $m = 1$ or $n = 1$.

Put $\log x = \mathrm{X}; \log y = \mathrm{Y}$.

In that case $xp = \mathrm{P}, yq = \mathrm{Q}$.

(ii) If ($z^k p$) and ($z^k q$) occur in the differential equations as in $\mathrm{F}(z^k p, z^k q) = 0$ or in $f_1(x, z^k p) = f_2(y, z^k q)$.

(a) Put $\mathrm{Z} = z^{k+1}$ if $k \neq -1$.

$$\frac{\partial Z}{\partial x} = (k + 1)z^k\frac{\partial z}{\partial x} = (k + 1)z^k p$$

$$\therefore\ z^k p = \frac{1}{k + 1}\frac{\partial Z}{\partial x}$$

Similarly, $z^k q = \frac{1}{k+1}\frac{\partial Z}{\partial y}$

(b) If $k = -1$, put $Z = \log z$

$$\frac{\partial Z}{\partial x} = \frac{p}{z}$$

Example 1:

Solve $p^2 + x^2y^2q^2 = x^2z^2$

The equation can be put in the form

$$z^2 = \left(\frac{p}{x}\right)^2 + (yq)^2$$

Put $\mathrm{X} = x^2, \mathrm{Y} = \log y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}2x; \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{1}{y}$$

i.e., $\frac{p}{x} = 2\mathrm{P}; yq = \mathrm{Q}$.

Hence the equation becomes $z^2 = 4\mathrm{P}^2 + \mathrm{Q}^2$.

This equation is of the form II (iii).

The solution of the equation is

$$(4 + a^2)(\log z)^2 = (\mathrm{X} + a\mathrm{Y} + b)^2$$

Hence the complete integral is

$$(4 + a^2)(\log z)^2 = (x^2 + a\log y + b)^2$$

Example 2:

Solve $p^2 + q^2 = z^2(x^2 + y^2)$.

The equation can be put in the form

$$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2 + y^2$$

Putting $\mathrm{Z} = \log z$, we get $p = z\frac{\partial Z}{\partial x}, q = z\frac{\partial Z}{\partial y}$.

The equation becomes

$$\left(\frac{\partial Z}{\partial x}\right)^2 + \left(\frac{\partial Z}{\partial y}\right)^2 = x^2 + y^2$$

i.e., $P^2 + Q^2 = x^2 + y^2$

The solution of the equation is

$$\mathrm{Z} = \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 + x^2} + \frac{y\sqrt{y^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + b$$

Hence the complete integral is obtained by putting log z instead of Z in the above solution.

Example 3:

Solve $z^4q^2 - z^2p = 1$

The equation can be put in the form $(z^2q)^2 - (z^2p) = 1$.

Put $z^3 = Z$.

$\therefore \frac{\partial Z}{\partial x} = 3z^2\frac{\partial z}{\partial x}$ and $\frac{\partial Z}{\partial y} = 3z^2\frac{\partial z}{\partial y}$.

Hence the given equation reduces to

$$\left(\frac{Q}{3}\right)^2 - \left(\frac{P}{3}\right) = 1$$

i.e., $Q^2 - 3P - 9 = 0$

It is of the form $F(P,Q) = 0$.

Hence its solution is

$$\mathrm{Z} = ax + by + c, \text{ where } b^2 - 3a - 9 = 0$$

$\therefore\ b = \pm\sqrt{3a+9}$

Hence the solution is $\mathrm{Z} = ax \pm \sqrt{3a+9} + c$.

Hence the complete integral is $z^3 = ax \pm \sqrt{3a+9} + c$.

Example 4:

Solve $\left(\frac{x}{p}\right)^n + \left(\frac{y}{q}\right)^n = z^n$.

Put $x^2 = \mathrm{X}, y^2 = \mathrm{Y}, z^2 = \mathrm{Z}$.

$$\frac{\partial z}{\partial x} = \frac{\partial\left(\mathrm{Z}^{1/2}\right)}{\partial \mathrm{X}}\cdot\frac{\partial \mathrm{X}}{\partial x}$$

$$= \frac{1}{2}\mathrm{Z}^{-1/2}\cdot\frac{\partial \mathrm{Z}}{\partial \mathrm{X}}\cdot 2x$$

$$= \frac{x}{z}\frac{\partial \mathrm{Z}}{\partial \mathrm{X}}$$

i.e., $\frac{x}{pz} = \frac{1}{\frac{\partial \mathrm{Z}}{\partial \mathrm{X}}} = \frac{1}{\mathrm{P}}$, where $\mathrm{P} = \frac{\partial \mathrm{Z}}{\partial \mathrm{X}}$.

Similarly $\frac{y}{qz} = \frac{1}{\frac{\partial Z}{\partial Y}} = \frac{1}{Q}$, where $Q = \frac{\partial Z}{\partial Y}$.

The equation reduces to $\frac{1}{\mathrm{P}^n} + \frac{1}{\mathrm{Q}^n} = 1$.

The complete integral is $\mathrm{Z} = a\mathrm{X} + b\mathrm{Y} + c$ where $\frac{1}{a^n} + \frac{1}{b^n} = 1$

i.e., $b = \dfrac{a}{(a^n-1)^{1/n}}$.

Hence the complete integral is

$$z^2 = ax^2 + \frac{a}{(a^n - 1)^{1/n}} y^2 + c$$

There is no singular integral, the general integral is got in the usual only.

Exercises 5:

1. $x^2p^2 + y^2q^2 = a^2$
2. $x^2p^2 + y^2q^2 = z$
3. $x^2p^2 + y^2q^2 = z^2$
4. $p^2 + q^2 = z^2(x + y)$
5. $z^2(p^2 + q^2) = x + y$

Study Learning Material Prepared by

Dr. S. KALAISELVI M.SC., M.Phil., B.Ed., Ph.D.,

Assistant Professor,

Department of Mathematics,

Sarah Tucker College (Autonomous),

Tirunelveli-627007.

Tamil Nadu, India.